On backward propagation of chaos

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$$X_t^{i,n} = \xi^i + \int_0^t b_u\Big(X_u^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_u^{j,n}}\Big)\, du + \int_0^t \sigma_u\Big(X_u^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_u^{j,n}}\Big)\, dW_u^i$$

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$$X^{i,n} \to X^i$$

with

$$X_t^i = \xi^i + \int_0^t b_u(X_u^i, \mathrm{law}(X_u^i))\, du + \int_0^t \sigma_u(X_u^i, \mathrm{law}(X_u^i))\, dW_u^i,$$

$\rightsquigarrow$ McKean-Vlasov SDE.

- Propagation of chaos: law$(X^{1,n}, \dots, X^{k,n}) \to$ law$(X^i)^{\otimes k}$
- Concentration: $P\Big(\mathcal{W}_2(\frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,n}}, \text{law}(X_t^i)) \geq x\Big) \leq 2e^{-Cnx^2}$
- Approximation and trend to equilibrium for nonlocal Fokker-Plank equations, e.g. $\partial_t \rho = \nabla \cdot (\nabla \rho + \rho \nabla W * \rho)$

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How about backward particles?

Digression:

Functional inequalities for BSDEs

joint with D. Bartl

On a Polish space (E, d), we consider the Wasserstein distance

$$\mathcal{W}_p^p(\mu,\nu) := \inf\left\{\iint_{E\times E} d^p(x,y)\,d\pi, \pi_1 = \mu;\ \ \mu_2 = \nu\right\}$$

and the Kullback-Leibler information divergence

$$H(\nu|\mu) := \begin{cases} E_\nu\left[\log \frac{d\nu}{d\mu}\right] & \text{if } \nu \ll \mu \\ +\infty & \text{else.} \end{cases}$$

- The probability measure μ satisfies $T_p(C)$ if

$$\mathcal{W}_p(\mu,\nu) \leq \sqrt{CH(\nu|\mu)} \quad \text{for all } \nu \in \mathcal{P}(E)$$

$\rightsquigarrow$ Talagrand (1996)

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Interesting application for us:

- Concentration of measure phenomenon $\rightsquigarrow$ Marton, Talagrand, Ledoux

$$\mu^{\otimes n}\Big(F - \int F d\mu^{\otimes n} \geq x\Big) \leq e^{-cx^2} \iff T_2(C)$$

cf. Gozlan (2009)

These inequalities are known to hold for various diffusion, including

- Brownian motion $\rightsquigarrow$ Feyel & Üstünel
- Stochastic differential equations $\rightsquigarrow$ Djelllout-Guillin-Wu; Üstuňel; Pal; Saussereau.

Theorem

Let $F : [0, T] \times \mathcal{C} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ *and* $G : \mathcal{C} \to \mathbb{R}^m$ *be Lipschitz continuous and let* (Y, Z) *satisfy*

$$dY_t = -F_t(Y_t, Z_t)\,dt + Z_t\,dW_t, \quad Y_T = G.$$

Then, the law of Y *satisfies* $T_2(C)$ *with* $C = 2(L_G + TL_F)^2 e^{2TL_F}$*.*

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- The constant C is optimal
- If $m = 1$; and $F_t(\omega, y, z) = F_t(z)$ is convex, it is enough to take $\mathrm{id} < F \leq$ quadratic, and $C = 2L_G$.

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- The constant C is optimal
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- Method:

  show that $Y = \varphi(W)$, with φ Lipschitz

  $\rightsquigarrow$ Ekren-Touzi-Zhang

Backward propagation of chaos

join with M. Laurière

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in (\mathbb{R}^m)^n$, put

$$L^n(\boldsymbol{x}) := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}.$$

- Let $(G^1, \ldots, G^n)$ be i.i.d. $\mathcal{F}_T$-measurable and consider the system

$$Y_t^{i,n} = G^i + \int_t^T F_u(Y_u^{i,n}, Z_u^{i,n}, L^n(\boldsymbol{Y}_u))\, du - \sum_{k=1}^{n} \int_t^T Z_u^{i,k,n}\, dW_u^k$$

where $(W^i, \ldots, W^n)$ are n independent Brownian motions.

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- Further consider the McKean-Vlasov BSDE

$$Y_t^i = G^i + \int_t^T F_u(Y_u^i, Z_u^i, \mathcal{L}(Y_u))\, du - \int_t^T Z_u^i\, dW_u^i.$$

Theorem

If $F : [0,T] \times \mathcal{C} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}^m$ *is Lipschitz continuous (in* (y, z, μ)*) and there is* $k > 2$ *such that* $E[|G|^k] < \infty$*, then*

$$\sup_t E\left[\mathcal{W}_2^2(L^n(\boldsymbol{Y}_t), \mathcal{L}(Y_t))\right] \leq C r_{n,m,k}$$

for some rate $r_{n,m,k} \downarrow 0$ *as* $n \to \infty$.

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If $k > m+5$ *and* $\sup_t E[|Z_t|^{2k}] < \infty$*, then*

$$E\left[\sup_t \mathcal{W}_2^2(L^n(\boldsymbol{Y}_t), \mathcal{L}(Y_t))\right] \leq C n^{-\frac{2}{m+8}}$$

$r_{n,m,k}$ is explicitly given, and depends on m and k.

Lemma

Lipschitz continuity yields

$$\mathcal{W}_2(L^n(\mathbf{Y}_t), \mathcal{L}(Y_t)) \leq e^{L_F T} \mathcal{W}_2(L^n(\tilde{\mathbf{Y}}_t), \mathcal{L}(Y_t))$$

where $\tilde{\mathbf{Y}} := (\tilde{Y}^1, \ldots, \tilde{Y}^n)$ *and* $(\tilde{Y}^1, \tilde{Z}^1), \ldots, (\tilde{Y}^n, \tilde{Z}^n)$ *are iid copies of* (Y, Z) *solving*

$$\tilde{Y}^i_t = G^i + \int_t^T F_u(\tilde{Y}^i_u, \tilde{Z}^i_u, \mathcal{L}(Y_u))\, du - \int_t^T \tilde{Z}^i_u\, dW^i_u.$$

- Use results by Fournier & Guillin (2015) and Horowitz & Karandikar (1994) to conclude.
- See also Sznitman.

Theorem

If $E[|G|^k] < \infty$ *for some* $k > 4$*, then*

$$P\Big(\mathcal{W}_2(L^n(\boldsymbol{Y}_t), \mathcal{L}(Y_t)) \geq x\Big) \leq C r_{n,x,k}$$

with $r_{n,x,k} \downarrow 0$ *(as* $n \to \infty$*) exponentially fast.*

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$$P\Big(\mathcal{W}_2(L^n(\mathbf{Y}_t), \mathcal{L}(Y_t)) \geq x\Big) \leq C r_{n,x,k}$$

with $r_{n,x,k} \downarrow 0$ (as $n \to \infty$) exponentially fast.

If F is Lipschitz in ω, and n large enough

$$P\Big(\mathcal{W}_{2,\mathcal{C}}(L^n(\mathbf{Y}), \mathcal{L}(Y)) \geq x\Big) \leq e^{-Cnx^2}$$

Proposition

If $E[|G|^k] < \infty$ *for some* $k > 2$*, then*

$$E\Big[\sup_t |Y_t^{i,n} - Y_t^i|^2 + \int_0^T |Z_t^{i,n} - Z_t^i|^2\, dt\Big] \le C r_{n,m,k}.$$

Proposition

If $E[|G|^k] < \infty$ for some $k > 2$, then

$$E\Big[\sup_t |Y_t^{i,n} - Y_t^i|^2 + \int_0^T |Z_t^{i,n} - Z_t^i|^2\, dt\Big] \leq Cr_{n,m,k}.$$

If the interaction is "linear", then $r_{n,m,k} = n^{-1}$, optimal, dimension-free rate! (see Buckdahn, Djehiche, Li & Peng (2009))

$\rightsquigarrow$

- Propagation of chaos: $\mathcal{W}_{2,\mathcal{C}}(\theta^{k,n}, \mathcal{L}(Y)^{\otimes k}) \leq kCr_{n,m,k}$ with $\theta^{n,k} := \text{law}(Y^{1,n}, \ldots, Y^{k,n})$.
- If the interaction is linear, then $r_{n,m,k} = n^{-1}$.

FBSDE: infinite dimensional PDEs and large population games

joint with M. Laurière

Now consider the system of FBSDEs

$$\begin{cases} dX_t^{i,n} = B_t(X_t^{i,n}, Y_t^{i,n}, L^n(\boldsymbol{X}_t, \boldsymbol{Y}_t))\,dt + \sigma\,dW_t^i \\ dY_t^{i,n} = -F_t(X_t^{i,n}, Y_t^{i,n}, Z_t^{i,n}, L^n(\boldsymbol{X}_t, \boldsymbol{Y}_t))\,dt + \sum_{k=1}^n Z_t^{i,k,n}\,dW_t^k \\ X_0^{i,n} = \xi^i \quad Y_T^{i,n} = G(X_T^{i,n}, L^n(X_T^{i,n})) \end{cases}$$

with B, F, and G Lipschitz continuous functions of linear growth,

$$|F_t(x, y, z, \mu)| \leq C(1 + |y| + |z| + M_2(\mu))$$

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with B, F, and G Lipschitz continuous functions of linear growth,

$$|F_t(x, y, z, \mu)| \leq C(1 + |y| + |z| + M_2(\mu))$$

and the Mckean-Vlasov FBSDE

$$\begin{cases} dX_t = B_t(X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + \sigma\, dW_t \\ Y_t = -F_t(X_t, Y_t, Z_t, \mathcal{L}(X_t, Y_t))\, dt + Z_t\, dW_t \\ X_t = \xi, \quad Y_T = G(X_T, \mathcal{L}(X_T)) \end{cases}$$

cf. Carmona & Delarue

Theorem

If there is $k > 2$ such that $E[|\xi|^k] < \infty$, then

$$E\left[\sup_t |X_t^{i,n} - X_t^i|^2 + |Y_t^{i,n} - Y_t^i|^2 + \int_0^T |Z_t^{i,i,n} - Z_t^i|^2\, dt\right] \leq C r_{n,m,l,k}$$

with $r_{n,m,l,k} \downarrow 0$ as $n \to \infty$.

Take $m = 1$. Let us consider the PDE

$$\begin{cases} \partial_t V(t,x,\mu) + B(x, V(t,x,\mu),\nu)\partial_x V(t,x,\mu) + \frac{1}{2}\mathrm{tr}\big(\partial_{xx} V(t,x,\mu)\sigma\sigma'\big) \\ \quad +F(x, V(t,x,\mu), \partial_x V(t,x,\mu)\sigma,\nu) \\ \quad + \int_{\mathbb{R}^d} \partial_\mu V(t,x,\mu)(y)\cdot B(y, V(t,x,\mu),\nu)d\mu(y) \\ \quad + \int_{\mathbb{R}^d} \frac{1}{2}\mathrm{tr}\left(\partial_y\partial_\mu V(t,x,\mu)(y)\sigma\sigma'\right)\, d\mu(y) = 0 \\ V(T,x,\mu) = G(x,\mu) \end{cases}$$

with $(t,x,\mu) \in [0,T) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. The derivative

$$\partial_\mu V(t,x,\mu)(y)$$

denotes the so-called L-derivative and ν is the law of $(\xi, U(t,\xi,\mu))$ when $\mathcal{L}(\xi) = \mu$.

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denotes the so-called L-derivative

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denotes the so-called L-derivative and ν is the law of $(\xi, V(t,\xi,\mu))$ when $\mathcal{L}(\xi) = \mu$.
$\rightsquigarrow$ Mou & Zhang (2019), Wu & Zhang (2019), Cardaliaguet et. al (2019), Chassagneux et. al (2015), Gangbo & Święch (2015)

$$\begin{cases} \partial_t v^{i,n}(t,\boldsymbol{x}) + B(x_i, v^{i,n}(t,\boldsymbol{x}), \frac{1}{n}\sum_{j=1}^n \delta_{(x_j, v^{j,n}(t,\boldsymbol{x}))})\partial_{x_i} v^{i,n}(t,\boldsymbol{x}) \\ \quad + \frac{1}{2}\mathrm{tr}\left(\partial_{x_i x_i} v^{i,n}(t,\boldsymbol{x})\sigma\sigma'\right) \\ \quad + F\left(x_i, v^{i,n}(t,\boldsymbol{x}), \partial_{x_i} v^{i,n}(t,\boldsymbol{x})\sigma(x_i), \frac{1}{n}\sum_{j=1}^n \delta_{(x_j, v^{j,n}(t,\boldsymbol{x}))}\right) = 0 \\ v^{i,n}(T,\boldsymbol{x}) = G\left(x_i, L^n(\boldsymbol{x})\right), \quad \boldsymbol{x} = (x_1,\dots,x_n) \in (\mathbb{R}^d)^n \\ i = 1,\dots,n. \end{cases}$$

with $(t, \boldsymbol{x}) \in [0, T] \times (\mathbb{R}^d)^n$

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with $(t,\boldsymbol{x}) \in [0,T] \times (\mathbb{R}^d)^n$

Theorem

If the master equation as a unique solution V such that $V(t,\xi,\mathcal{L}(\xi)) = Y^{t,\xi}$ for all $\mathcal{F}_t$-measurable $\xi \in L^2(P)$, then

$$E\left[|v^{1,n}(t,\xi_1,\ldots,\xi_n) - V(t,\xi_1,\mu)|^2\right] \le Cr_{n,l,k,2}$$

for all t and all i.i.d. ξ^i with law μ.

Idea

- Chassagneux; Crisan & Delarue (2015) recently gave conditions under which
$$V(t,\xi,\mathcal{L}(\xi)) = Y_t^{t,\xi}$$

- by classical FBSDE theory (e.g. Ma; Protter & Young)
$$v^{1,n}(t,\xi_1,\dots,\xi_n) = Y_t^{t,\xi^1,\dots,\xi^n}$$

- thus, suffices to show $Y_t^{t,\xi^1,\dots,\xi^n} \to Y_t^{t,\xi}$.

PDE-based approach to a similar result $\rightsquigarrow$ Cardaliaguet, Delarue, Lasry & Lions (2019)

Consider $dX_t^i = \alpha_t^i + \frac{1}{n}\sum_{j=1}^n \alpha_t^j\, dt + \sigma\, dW_t^i$ and

$$J^i(\alpha) := E\Big[|X_T^i|^2 + \int_0^T \frac{1}{2}(\alpha_t^i)^2 + (\frac{1}{n}\sum_{j=1}^n \alpha_t^j)^2\, dt\Big] \to \min$$

Pontryagin for N-player game: $\hat{\alpha}^{i,n}$ n-Nash $\rightsquigarrow \hat{\alpha}^{i,n} = -Y^{ii} + R_n(Y^{ij})_{ij}$
with

$$\begin{cases} dX_t^i = \hat{\alpha}_t^{i,n} + \frac{1}{n}\sum_{j=1}^n \hat{\alpha}_t^{j,n}\, dt + \sigma\, dW_t^i \\ dY_t^{ij} = \sum_{k=1}^n Z_t^{ijk}\, dW_t^k, \quad Y_T^{ij} = 2\delta_{ij} X_T^i \end{cases}$$

and $R_n(Y^{ij})_{ij=1,\dots,n} \to 0$

Let $dX_t = (\alpha_t + E[\alpha_t])\,dt + \sigma\,dW_t$ and a flow of measures μ. Find $\hat{\alpha}$ s.t.

$$\inf_{\alpha} E\Big[|X_T^{\alpha}|^2 + \int_0^T \frac{1}{2}\alpha_t^2 + \Big(\int x\,d\mu_t(x)\Big)^2\,dt\Big] \quad \text{and } \mu_t = \mathrm{law}(\hat{\alpha}_t^{\mu}).$$

Pontryagin for extended MFG: $\hat{\alpha} = -Y$ with

$$\begin{cases} dX_t = -(Y_t + E[Y_t])\,dt + \sigma\,dW_t \\ dY_t = Z_t\,dW_t, \quad Y_T = 2X_T \end{cases}$$

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$$\begin{cases} dX_t = -(Y_t + E[Y_t])\,dt + \sigma\,dW_t \\ dY_t = Z_t\,dW_t, \quad Y_T = 2X_T \end{cases}$$

By the above results:

$$\hat{\alpha}^{i,n} \to \hat{\alpha}$$

See also works by Cardaliaguet, Delarue, Lasry & Lions (2009), Lacker (2016, 2017, 2018), Fischer (2017), Nutz, San Martin & Tan (2018)

Summary

- Functional inequalities for BSDE
- Backward propagation of chaos
  - explicit convergence rates
  - concentration inequalities
- Forward backward "particles"
  - approximation of master equation
  - convergence to extended MFG

Thank You!