# Global Well-posedness of Non-Markovian Multidimensional Superquadratic BSDE

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- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a $n$-dimensional Brownian motion $W$.
- $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ is the augmented filtration generated by a stochastic process $X$. In particular, we will denote $\mathbb{F} := \mathbb{F}^W$ unless otherwise stated.
- The time horizon of interest is given by $[0, T]$.
- For a stochastic process $X$, we will denote $X_t(\omega)$ be the time $t$-value of $X$ for realization $\omega$. On the other hand, $X_{[0,t]}(\omega)$ is the realized path of $X$ from time 0 to $t$. As always in probability theory, we will omit the dependency in $\omega$ unless it is needed.
- A vector in $\mathbb{R}^d$ is considered as a matrix in $\mathbb{R}^{d\times 1}$.
- The norm of matrix is given by Frobenius norm which is denoted by $|\cdot|$, that is,

$$|A| := \sqrt{\operatorname{tr}(AA^\intercal)}$$

Let $(W, \mathbb{F}^W := (\mathcal{F}^W_t)_{t\in[0,T]}, \mathbb{P})$ be a $n$-dim. Brownian motion. The most classical form of backward stochastic differential equation BSDE$(\Xi, F)$ is

$$Y_t = \Xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

| Input | Output | |
|---|---|---|
| $\Xi \in \mathcal{F}_T$ | $Y$ | : $\mathbb{R}^d$-valued adapted |
| $F \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+d\times n})$ | $Z$ | : $\mathbb{R}^{d\times n}$-valued adapted |

**Remark**

We have $Z$ as a part of solution because we want $Y$ to be adapted. For example, consider $dY_t = 0; \quad Y_T = \Xi$. Then, $Y_t = \Xi$ is not adapted.

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**Comparison Principle**

Let $d = 1$. If $\Xi \leq \tilde{\Xi}$ and $F(s, y, z) \leq \tilde{F}(s, y, z)$, then $Y_t \leq \tilde{Y}_t$ for all $t$ almost surely, where $Y$ and $\tilde{Y}$ are the solutions of BSDE$(\Xi, F)$ and BSDE$(\tilde{\Xi}, \tilde{F})$

- $F$ is linear (Bismut, 1973)
  $\Rightarrow$ Dual approach to stochastic control problem
- **$F$ is Lipschitz (Pardoux and Peng, 1990)**
- **$d = 1$ and $F(s, y, z)$ is quadratic growth in $z$**
  **Kobylanski (2000) and others.**
  $\Rightarrow$ Risk-sensitive control problem
- $d > 1$ and $F(s, y, z)$ is quadratic growth in $z$
  - forward process coupled to BSDE: FBSDE (see works of Ma, Zhang and others)
  - involving backward stochastic integral: BDSDE $\Rightarrow$ Feynman-Kac for SPDE
  - $\cdots$

- Agent 1 controls $\alpha^1$ and agent 2 controls $\alpha^2$. The $\alpha = (\alpha^1, \alpha^2)$ requires to be bounded predictable process.
- State process: for $b(\alpha) = p^1\alpha^1 + p^2\alpha^2$ and a constant $\sigma > 0$,

$$dX_t = \sigma dW_t = b(\alpha_t)dt + \sigma dW_t^\alpha$$

where $W^\alpha = W_t - \int_0^t \sigma^{-1} b(\alpha_s)ds$ is a Brownian motion under $\mathbb{P}^\alpha$ where

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \mathcal{E}\left(\int \sigma^{-1} b(\alpha_s)dW_s\right)_T$$

- Cost Functionals: for agent $i$,

$$\mathbb{E}^{\mathbb{P}^\alpha}\left[\int_0^T \frac{\gamma^i}{2}|\alpha^i|^2 ds + g^i(X_{[0,T]})\right]$$

where $\gamma^i > 0$.

Let $H^i(z,\alpha) = z^i b(\alpha) + \frac{\gamma^i}{2}|\alpha^i|^2$.

1. Find $\hat{\alpha}(z)$ such that

$$H^1(z,\hat{\alpha}^1,\hat{\alpha}^2) \le H^1(z,\alpha^1,\hat{\alpha}^2), \quad H^2(z,\hat{\alpha}^1,\hat{\alpha}^2) \le H^2(z,\hat{\alpha}^1,\alpha^2)$$

2. Find a pair of adapted solution $(Y,Z)$ that satsfies $Y_T = g(X_{[0,T]})$ for $X_t = X_0 + \sigma W_t$ and

$$Y_s = -H(Z_s\sigma^{-1},\hat{\alpha}(Z_s\sigma^{-1}))ds + Z_s dW_s.$$

   Note the driver $H^i(Z_s\sigma^{-1},\hat{\alpha}(Z_s\sigma^{-1}))$ is

$$Z_t^i\sigma^{-2}\left(\frac{|p^1|^2 Z_t^1}{\gamma^1} + \frac{|p^2|^2 Z_t^2}{\gamma^2}\right) + \frac{p_i}{2\gamma^i|\sigma|^2}|Z_t^i|^2$$

3. $\hat{\alpha}^i(Z_t\sigma^{-1})$ is an optimal control for agent $i$ if bounded.

$$
\begin{aligned}
Y_t^{i,\alpha^1,\alpha^2} &= g^i(X_{[0,T]}) + \int_t^T H^i(Z_s\sigma^{-1}, \alpha_s^1, \alpha_s^2)ds - \int_t^T Z_s^i dW_s \\
&= g^i(X_{[0,T]}) + \int_t^T \frac{\gamma^i}{2}|\alpha_s^i|^2 ds - \int_t^T Z_s^i dW_s^\alpha
\end{aligned}
$$

Therefore, $Y_0^{i,\alpha^1,\alpha^2} = \mathbb{E}^{\mathbb{P}^\alpha}\left[g^i(X_{[0,T]}) + \int_t^T \frac{\gamma^i}{2}|\alpha_s^i|^2 ds\right]$. Moreover, by comparison theorem, we have

$$
\begin{aligned}
Y_0^{1,\hat{\alpha}^1,\hat{\alpha}^2} &\leq Y_0^{1,\alpha^1,\hat{\alpha}^2} \\
Y_0^{2,\hat{\alpha}^1,\hat{\alpha}^2} &\leq Y_0^{2,\hat{\alpha}^1,\alpha^2}
\end{aligned}
$$

By exponential change of variable, we can remove $\frac{p_i}{2\gamma^i|\sigma|^2}|Z^i|^2$ term. For "Lipschitz" $f$ and locally Lipschitz $g$, we consider the existence and uniqueness of bounded solution for the following BSDE: $Y_T = \xi(W_{[0,T]})$ and

$$Y_s = -\left(f(Z_s) + Z_s g(Z_s)\right) ds + Z_s dW_s$$

Here, $\xi$ and $f$ are $\mathbb{R}^d$-valued and $g$ is $\mathbb{R}^n$-valued with $d \geq 1$. The coefficients $\xi$ depend on the path of $W$, making the BSDE non-Markovian and it is Lipschitz wrt sup norm on $C([0,T];\mathbb{R}^n)$.

- Bounding $Z$ using Malliavin calculus can prove the local existence and uniqueness of solution, but not the global existence due to the explosion of the bound in backward iteration.
- Result: If $\xi, f, g$ is stable under the perturbation of $W$, then there exists a unique bounded solution.

# Previous literature

| | Non-Markovian | zg(z) term | General DQ term | Large Coefficients | W-irregularity |
|---|---|---|---|---|---|
| Tevzadze (2008) | Yes | Yes | Yes | No | Yes |
| Cheridito and N. (2015) | No | Yes | No | Yes | No |
| Hu and Tang (2016) | Yes | No | Yes | Yes | Yes |
| Xing and Zitkovic (2018) | No | Yes | Yes | Yes | No |
| Harter and Richou (2019) | Connecting Tevzadze — Hu and Tang | | | | |
| **Presentation (2020)** | Yes | Yes | No | Yes | No |

- Examples: Kramkov and Pulido (2016)$\cdots$
- Counterex. with $W$-Lipschitzness: Chang et al. (1992).
- **Markovian+ Special Structure**: Cheridito and N. (2015) $\subset$ Xing and Žitković (2018)
  $\Rightarrow$ Based on PDE technique: unable to generalize to non-Markovian case unless DQ.

Assume the following conditions: There exist positive constants $C, K$ and an increasing function $l : \mathbb{R}_+ \to \mathbb{R}_+$,

(H1) $\xi : C([0,T];\mathbb{R}^n) \to \mathbb{R}^d$ satisfies $|\xi(\omega)|^2 \leq C$ and $|\xi(\omega) - \xi(\omega')|^2 \leq K \|\omega - \omega'\|_{sup}^2$. for all $\omega, \omega' \in C([0,T];\mathbb{R}^n)$.

(H2) $f : [0,T] \times C([0,T];\mathbb{R}^n) \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^d$ and $g : [0,T] \times C([0,T];\mathbb{R}^n) \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^n$satisfies

- $\left|f(s, x_{[0,s]}, 0, 0)\right|^2 \leq C$ and $\left|g(s, x_{[0,s]}, 0, 0)\right|^2 \leq C$.
-

$$
\begin{aligned}
&|f(s,\omega,y,z) - f(s,\omega',y',z')|^2 \\
&\qquad \leq C\left(\sup_{r\in[0,s]} |\omega_r - \omega'_r|^2 + |y-y'|^2 + |z-z'|^2\right) \\
&|g(s,\omega,y,z) - g(s,\omega,y',z')|^2 \\
&\qquad \leq l(|z|+|z'|)\left(\sup_{r\in[0,s]} |\omega_r - \omega'_r|^2 + |y-y'|^2 + |z-z'|^2\right)
\end{aligned}
$$

## Theorem

*Then, the BSDE*

$$Y_t = \xi(W_{[0,T]}) + \int_t^T \big(f(s, W_{[0,s]}, Y_s, Z_s) + Z_s g(s, W_{[0,s]}, Y_s, Z_s)\big)\, ds - \int_t^T Z_s dW_s$$

*has a unique solution* $(X, Y, Z) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2$ *such that* $(Y, Z)$ *is bounded. In particular, there is a continuous and bounded function* $k : [0,T] \times C([0,T]; \mathbb{R}^n) \to \mathbb{R}^d$ *such that* $Y_t = k(t, W_{[0,t]})$ *and*

$$\left| k(t, x_{[0,t]}) - k(t, x'_{[0,t]}) \right|^2 \leq \rho(T-t) \sup_{s \in [0,t]} |x_s - x'_s|^2$$

*where* $\rho(x) = \left(K + \frac{C}{2(C+1)}\right) e^{2(C+1)x} - \frac{C}{2(C+1)}$*. In addition,* $|Z_t| \leq \sqrt{\rho(T-t)}$ *in* $dt \otimes d\mathbb{P}$*-almost everywhere sense.*

- The conditions can be weakened so that Malliavin derivatives are bounded almost surely almost everywhere.
- Bounded assumption may be weakened as well.
- We can add diagonally quadratic term $a^i|z^i|^2$ to $f^i$. However, general diagonally quadratic $f$ case is still under investigation.
- If the terminal conditions and the driver approximate general quadratic BSDE with error being small enough, the existence still holds: using Tevzadze (2008).

- Let $\Phi$ be the unique solution of

$$d\Phi_t = b_f(t, \Phi_{[0,t]}, M_{[0,t]})dt + \sigma_f(t)dM_t, \qquad \Phi_0 \in \mathbb{R}^m, t \in [0,T].$$

  where $b_f(y, \phi, m)$ is Lipschitz in $(\phi, m)$ wrt sup norm and $\sigma_f \in \mathcal{C}^1$. Then, the map $\mathcal{S} : M \mapsto \Phi$ satisfies Lipschitz property wrt sup norm.
- Reflection also preserves Lipschitzness wrt $W$ if the boundary is good enough and the reflection direction is well-defined.[1]. Therefore, reflected SDE and SDE driven by reflected Brownian motion is Lipschitz wrt $W$ if the drift and the volatility satisfies the condition in the first bullet point.

[1] Dupuis and Ishii (1991)

$$Y_t = \xi(W_{[0,T]}) + \int_t^T f(Z_s) + Z_s g(Z_s) ds - \int_t^T Z_s dW_s$$

$$Y_t = \xi\left(\left(W - \int_0^\cdot g(Z_s)ds\right)_{[0,T]} + \left(\int_0^\cdot g(Z_s)ds\right)_{[0,T]}\right) + \int_t^T f(Z_s)ds - \int_t^T Z_s(dW_s - g(Z_s)ds)$$

Under changed measure that make $W - \int_0^\cdot g(Z_s)ds$ a Brownian motion, the $Zg(Z)$ term can be removed from the driver and absorbed to terminal condition.

Strategy: Solve the second equation and transform to the first equation.

- In general, the filtration generated by Girsanov transformed process is smaller than the filtration generated by the original process. Therefore, the solution is a weak solution.
- The filtration problem can be resolved if the corresponding FBSDE has a solution adapted to the filtration generated by the forward process:

$$X_t = W_t - \int_0^t g(Z_s)ds$$

$$Y_t = \xi(X_{[0,T]}) + \int_t^T f(Z_s)ds - \int_t^T Z_s dW_s.$$

Unfortunately, the result on such FBSDE is limited. Hu (2019) studied the case under different conditions (only local solution in time).

- Localise $g$ in $Z$ so that $g$ is Lipschitz.
- Show the existence and uniqueness of solution for FBSDE on $u \in [T-\varepsilon, T]$: typically easy. Analyse to conclude the solution is adapted to the forward process (decoupling field) and $Y$ is Lipschitz in the perturbation of $X$.

$$
\begin{aligned}
dX_t^{u,x_{[0,u]}} &= g(t, x_{[0,u]} \otimes X_{[u,t]}^{u,x_{[0,u]}}, Y_t^{u,x_{[0,u]}}, Z_t^{u,x_{[0,u]}})dt + dW_t \\
dY_t^{u,x_{[0,u]}} &= -f(t, x_{[0,u]} \otimes X_{[u,t]}^{u,x_{[0,u]}}, Y_t^{u,x_{[0,u]}}, Z_t^{u,x_{[0,u]}})dt + Z_t^{u,x_{[0,u]}} dW_t \\
X_s^{u,x_{[0,u]}} &= x_s \text{ for } s \leq u \\
Y_T^{u,x_{[0,u]}} &= \xi(x_{[0,u]} \otimes X_{[u,T]}^{u,x_{[0,u]}}) \\
k(t, x_{[0,t]}) &:= Y_t^{t,x_{[0,t]}} \\
Y_t^{u,x_{[0,u]}} &= k(t, X_{[0,t]}^{u,x_{[0,u]}}).
\end{aligned}
$$

- Perform Girsanov transform to conclude the local existence of a unique solution for BSDE. Use the estimation from FBSDE to conclude $Y_{T-\varepsilon}$ again satisfies the assumption (H1). Estimate $Z$ using vertical derivative of $Y$ with respect to $W$ (functional Ito calculus).
- Repeat the argument until reach 0.
- Un-localise $g$ since the bound of $Z$ does not depend on the Lipschitz coefficient of $g$.

## Theorem

*Let* $Y_t = y(t, W_{[0,t]})$ *and* $Z_t = z(t, W_{[0,t]})$ *be a solution of the BSDE*

$$Y_s = -f(s, W_{[0,s]}, Y_s, Z_s) - Z_s g(s, W_{[0,s]}, Y_s, Z_s)ds + Z_s dW_s.$$

*with* $Y_T = \xi(W_{[0,T]})$. *Assume that the SDE*

$$dP_t = g(s, P_{[0,s]}, y(s, P_{[0,s]}), z(s, P_{[0,s]}))ds + dW_s; \qquad P_0 = 0$$

*has a unique (strong) solution* $P$ *and that* $g(\cdot, P_{[0,\cdot]}, y(\cdot, P_{[0,\cdot]}), z(\cdot, P_{[0,\cdot]})) \in \mathbb{H}^{BMO}$. *Then, FBSDE*

$$\begin{aligned} dP_t &= g(s, P_{[0,s]}, Q_s, R_s)ds + dW_s, & P_0 &= 0 \\ dQ_s &= -f(s, P_{[0,s]}, Q_s, R_s)ds + R_s dW_s & Q_T &= \xi(P_{[0,T]}) \end{aligned}$$

*has a solution* $Q_s = y(s, P_{[0,s]}), R_s = z(s, P_{[0,s]})$.

- FBSDE: strong formulation of stochastic differential game.
- The continuity with respect to $P$ can be relaxed to measurable condition in some cases: in particular, when the system is Markovian. (SDE strong well-posedness result with measruable coefficients such as Zvonkin's)
- Combination with Xing and Žitković (2018) on Markovian BSDE provides interesting results on FBSDEs.

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