Time Consistency in Decision Making

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Mathematical Finance Colloquium

University of Southern California

November 13, 2017

Let $\mathcal{X}$ be a non-empty set, and assume that at each time $t \in \{0, 1, \dots, T\}$ we have **a total preorder $\preccurlyeq_t$** on $\mathcal{X}$, that admits a *numerical representation* $\varphi_t : \mathcal{X} \to [-\infty, \infty]$, i.e.

$$x \preccurlyeq_t y \quad \Longleftrightarrow \quad \varphi_t(x) \le \varphi_t(y), \qquad x, y \in \mathcal{X}.$$

Main Question:

If the decisions are made through time using the preorder $\preccurlyeq_t$ or its numerical representation φ_t, $t = 0, 1, 2, \dots, T$, how to insure that the decisions are made **consistently in time**?

A total preorder is a binary relation $\preccurlyeq$ on $\mathcal{X}$ such that

1. (reflexive) $x \preccurlyeq x$, for any $x \in \mathcal{X}$;
2. (transitive) if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$;
3. (total) for any $x, y \in \mathcal{X}$, $x \preccurlyeq y$ or $y \preccurlyeq x$,

The progressive assessment of preferences should be an integral part of the decision making process.

The assessment of preferences should be done in such a way that the **future preferences are assessed consistently with the present ones**.

Main Goal:

To develop a **unified theory for studying time consistency** for dynamic risk and dynamic performance measures.

We will take a top-down approach and define time consistency for functions φ that are only *local and monotone.*

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We will take a top-down approach and define time consistency for functions φ that are only *local and monotone.*

1. T.R. Bielecki, I. Cialenco, M. Pitera, *A unified approach to time consistency of dynamic risk measures and dynamic performance measures in discrete time*, Forthcoming in Mathematics of Operations Research (28 pages), 2017.
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- T.R. Bielecki, I. Cialenco, S. Drapeau and M. Karliczek, *Dynamic Assessment Indices*, Stochastics, vol. 88, No 1, pp. 1-44, 2016.
- T.R. Bielecki, I. Cialenco, I. Iyigunler and R. Rodriguez, *Dynamic Conic Finance: Pricing and Hedging in Market Models with Transaction Costs via Dynamic Coherent Acceptability Indices*, IJTAF, vol. 16, No 1, 2013.
- T.R. Bielecki, I. Cialenco and Z. Zhang, *Dynamic Coherent Acceptability Indices and their Applications to Finance*, Math. Fin., Vol. 24(3), pp. 411-441, July 2014 (accepted 2011).
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Literature review

Before the theory of risk measures:

- Koopmans [Koo60] - Stationary ordinal utility and impatience
- Kreps and Porteus [KP78] - Temporal resolution of uncertainty and dynamic choice theory
- Duffie and Epstein [DE92] - Stochastic differential utility
- Sarin and Wakker [SW98] - dynamic consistency for non-Expected utilities in a decision theoretic framework

There exists a significant literature on dynamic risk and performance measures, with *'time consistency axiom'* playing a crucial role.

"The dynamic consistency axiom turns out to be the heart of the matter."

A. Jobert and L. C. G. Rogers

Valuations and dynamic convex risk measures, Math Fin 18(1), 2008, 1-22.

Notations

- T-fixed time horizon; Discrete time setup $\mathcal{T} := \{0, 1, \dots, T\}$;
- $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ - the underlying filtered probability space;
- $L^p(\mathcal{F}_t) := L^p(\Omega, \mathcal{F}_t, \mathbb{P};\ \mathbb{R})$, $p \in \{0, 1, \infty\}$,

  $\bar{L}^p_t := L^p(\Omega, \mathcal{F}_t, \mathbb{P};\ \overline{\mathbb{R}})$,
- $\infty - \infty = -\infty \quad \text{and} \quad 0 \cdot \infty = 0$,
  $E[X|\mathcal{F}_t] := \lim_{n\to\infty} E[(X^+ \wedge n)|\mathcal{F}_t] - \lim_{n\to\infty} E[(X^- \wedge n)|\mathcal{F}_t]$.

Case of Random Variables: $\mathcal{X} = L^p(\mathcal{F}_T)$

$X \in L^p(\mathcal{F}_T)$ is interpreted as terminal payoff, or portfolio's value at T.
In this talk, we focus on this case.

Throughout we will assume *zero interest rates*.

Case of Stochastic Processes was studied too.
$\mathcal{X} = \mathbb{V}^p = \{(V_t)_{t\in\mathbb{T}} : V_t \in L_t^p\}$.
$V \in \mathbb{V}^p$ is interpreted as a cash flow with dividend payments V_t at $t \in \mathcal{T}$.

Preference orders on $\mathcal{X}$ Dynamic LM-measures approach

Definition

A **dynamic LM-measures** is a family $\{\varphi_t\}_{t\in\mathbb{T}}$ of maps $\varphi_t : \mathcal{X} \to \bar{L}^0_t$, such that, for any $t \in \mathbb{T}$, φ_t is

- **Local:** $1_A\varphi_t(X) = 1_A\varphi_t(1_A X)$, for any $X \in \mathcal{X},\ A \in \mathscr{F}_t$
- **Monotone:** $X \leq Y \Rightarrow \varphi_t(X) \leq \varphi_t(Y)$, for any $X, Y \in \mathcal{X}$.

- Two minimal properties, with clear financial interpretation, to define a preference order $\mathcal{X}$.
- Locality is also known as regularity, zero-one law, or relevance property.
- With some additional properties, an LM-measure becomes a monetary utility measure, or a risk measure, or a performance measure, or assessment index, etc.

Examples of LM measures

- Conditional essential infimum
  $\varphi_t(X) = \operatorname{ess\,inf}_t X$;
- $\varphi_t(X) = E[X \mid \mathscr{F}_t]$;
- Value At Risk
  $\varphi_t(X) = -\mathrm{V@R}_\alpha(X \mid \mathscr{F}_t) = \operatorname{ess\,sup}\{s \in L^0(\mathscr{G}) \mid P(X < s|\mathscr{G}) \leq \alpha\}$;
- Average Value at Risk
  $\varphi_t(X) = -\int_0^\alpha \mathrm{V@R}_\beta(X \mid \mathscr{F}_t) d\beta$;
- Gain to Loss Ratio
  $\varphi_t(X) = \frac{E[X \mid \mathscr{F}_t]}{E[X^- \mid \mathscr{F}_t]}$, if $E[X \mid \mathscr{F}_t] > 0$ and zero otherwise.
- Sortino Ratio $\varphi_t(X) = \frac{E[X \mid \mathscr{F}_t]}{\sqrt{E[(X^-)^2 \mid \mathscr{F}_t]}}$;
- Any dynamic risk measure; any dynamic assessment index.

Existing approaches to time consistency

Generic approach: families of benchmark sets Tutsch [Tut08].
It is implied by the proposed methodology, but not equivalent.

Idiosyncratic approaches: define time consistency in terms of objects specific to a certain class of measures

- Conditional acceptance sets
- Dynamic of the minimal penalty functions
- Cocycle condition
- g-expectation
- Recursive construction
- rectangular property, prudence, etc

All these approaches are particular cases of the proposed method; see the survey [BCP16]

Update Rules Approach to Time Consistency

Definition

A family $\mu = \{\mu_{t,s} : t, s \in \mathbb{T}, t < s\}$ of maps

$$\mu_{t,s} : \bar{L}^0_s \to \bar{L}^0_t$$

is called an **update rule** if μ satisfies the following conditions:

1) **(Local)** $\mathbb{1}_A \mu_{t,s}(m) = \mathbb{1}_A \mu_{t,s}(\mathbb{1}_A m)$;

2) **(Monotone)** if $m \geq m'$, then $\mu_{t,s}(m) \geq \mu_{t,s}(m')$;

for any $s > t$, $A \in \mathscr{F}_t$ and $m, m' \in \bar{L}^0_s$.

An update rule is meant to relate the assessment level of preferences between different times.

Definition

The update rule μ is called

- s-**invariant** if $\mu_{t,s}(m) = \mu_t(m)$, for $s \geq t$, $m \in \bar{L}^0_s$.
- **projective** if it is s-invariant and $\mu_t(m_t) = m_t$, for $t \in \mathbb{T}$, $m_t \in \bar{L}^0_t$.

Example

The families $\mu^1 = \{\mu^1_t\}_{t\in\mathbb{T}}$ and $\mu^2 = \{\mu^2_t\}_{t\in\mathbb{T}}$ given by

$$\mu^1_t(m) = E[m|\mathscr{F}_t], \quad \text{and} \quad \mu^2_t(m) = \operatorname{ess\,inf}_t m, \quad m \in \bar{L}^0,$$

are projective update rules.

For a fixed $\varepsilon \in (0,1)$, consider the update rule (not s-invariant)

$$\mu^3_{t,s}(m, X) = \begin{cases} \varepsilon^{s-t} E[m|\mathcal{F}_t], & \text{on } \{E[m|\mathcal{F}_t] \geq 0\}, \\ \varepsilon^{t-s} E[m|\mathcal{F}_t], & \text{on } \{E[m|\mathcal{F}_t] < 0\}. \end{cases} .$$

Definition

Dynamic LM-measure φ is **μ-acceptance time consistent** if

$$\varphi_s(X) \geq m_s \quad \Longrightarrow \quad \varphi_t(X) \geq \mu_{t,s}(m_s),$$

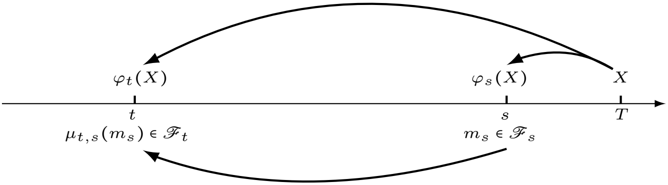
for all $s > t$, $s, t \in \mathbb{T}$, $X \in \mathcal{X}$, and $m_s \in \bar{L}^0_s$.

Similarly, φ is **μ-rejection time consistent** if

$$\varphi_s(X) \leq m_s \quad \Longrightarrow \quad \varphi_t(X) \leq \mu_{t,s}(m_s),$$

for all $s > t$, $s, t \in \mathbb{T}$, $X \in \mathcal{X}$, and $m_s \in \bar{L}^0_s$.

In this talk we focus on acceptance time consistency.



An analog of "Dynamic Programming Principle"

Proposition

The LM-measure φ is μ-acceptance time consistent if and only if

$$\varphi_t(X) \geq \mu_{t,s}(\varphi_s(X)),$$

for any $X \in \mathcal{X}$ and $s > t$.

Proof.

($\Rightarrow$) Take in the definition of acceptance time consistency $m_s = \varphi_s(X)$.

($\Leftarrow$) Let $m_s \in \bar{L}^0_s$ be such that $\varphi_s(X) \geq m_s$. By monotonicity of μ,

$$\varphi_t(X) \geq \mu_{t,s}(\varphi_s(X)) \geq \mu_{t,s}(m_s).$$

□

Strong Time Consistency

Definition

A dynamic LM-measure φ is **strongly time consistent** if there exist an update rule μ such that φ is both μ-acceptance and μ-rejection time consistent.

Proposition

The following statements are equivalent:

1) *φ is strongly time consistent.*

2) *There exists an update rule μ such that*
$$\mu_{t,s}(\varphi_s(X)) = \varphi_t(X), \quad X \in L^p_T,\ s > t.$$

3) *There exists a one-step update rule μ such that*
$$\mu_{t,t+1}(\varphi_{t+1}(X)) = \varphi_t(X), \quad X \in L^p,\ t < T.$$

4) *For any $X, Y \in L^p$ and $s > t$,*
$$\varphi_s(X) = \varphi_s(Y) \quad \Longrightarrow \quad \varphi_t(X) = \varphi_t(Y).$$

- **Strong time consistency** is one of the most popular forms of time consistency. Suitable, and traditionally used, for dynamic convex risk measures.

- **A dynamic monetary utility measure** is an LM-measure that is also

  normalized $\varphi_t(0) = 0$, and
  cash-additive $\varphi_t(X + m_t) = \varphi_t(X) + m$, $m \in L^p_t$.

- **A monetary risk measure** is the negative of a monetary utility measure.

- A dynamic monetary utility measure φ is **representable**, if

$$\varphi_t(X) = \operatorname*{ess\,inf}_{Q \prec P} \Big(E_Q[X \mid \mathcal{F}_t] + \alpha_t^{\min}(Q) \Big), \quad X \in \mathcal{X},$$

  where $\alpha_t^{\min}$ is a 'minimal penalty function'.

  This type of representation is called robust or numerical representations.

Proposition

Let φ be a representable dynamic monetary utility measure on L^∞. The following properties are equivalent:

1) φ *is strongly time consistent.*
2) φ *is* **recursive**, *i.e. for any* $X \in L^p$, $s, t \in \mathbb{T}$, $s > t$,

$$\varphi_t(X) = \varphi_t(\varphi_s(X)).$$

3) $\mathcal{A}_t = \mathcal{A}_{t,s} + \mathcal{A}_s$, *for all* $t, s \in \mathbb{T}$, $s > t$, $\mathcal{A}_{t,s} := \{X \in L^p \cap \bar{L}^0_s : \varphi_t(X) \geq 0\}$.
4) *For any* $Q \prec P$, $t, s \in \mathbb{T}$, $s > t$,

$$\alpha_t^{\min}(Q) = \alpha_{t,s}^{\min}(Q) + E_Q[\alpha_s^{\min}(Q) \mid \mathcal{F}_t].$$

5) *For any* $X \in L^p$, $Q \prec P$, $s, t \in \mathbb{T}$, $s > t$,

$$\varphi_t(X) - \alpha_t^{\min}(Q) \leq E_Q[\varphi_s(X) - \alpha_s^{\min}(Q) \mid \mathcal{F}_t].$$

Recursivity plays a critical role in pricing by using risk measures.

Example (Dynamic Entropic Utility/Risk Measure)

$$\varphi_t^\gamma(X) = \begin{cases} \frac{1}{\gamma} \ln E[\exp(\gamma X)|\mathcal{F}_t] & \text{if } \gamma \neq 0, \\ E[X|\mathcal{F}_t] & \text{if } \gamma = 0, \end{cases}$$

where $X \in \mathcal{X} = L^\infty$, $t \in \mathbb{T}$; $\theta = -\gamma$ is the risk-aversion parameter

- For $\gamma \leq 0$, the map φ_t^γ is a dynamic concave utility measure.
- For any $\gamma \in \mathbb{R}$, the map φ^γ is strongly time consistent.
- More generally, for $U : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ a strictly increasing, continuous function, a **dynamic certainty equivalent** is defined as

$$\varphi_t(X) = U^{-1}(E[U(X)|\mathcal{F}_t]), \qquad X \in \mathcal{X},\ t \in \mathbb{T}. \tag{5.1}$$

- φ is a strongly time consistent dynamic LM-measure.
- Every dynamic LM-measure, which is finite, normalized, strictly monotone, continuous, law invariant, admits the Fatou property, and is strongly time consistent, can be represented as (5.1) for some U [KS09].
  See also [BBN14, BCDK16].

- $\mathrm{V@R}_\alpha$ is *not* strongly time consistent [CS09].
- Dynamic Gain Loss Ratio (dGLR) defined as

$$\varphi_t(X) := \begin{cases} \frac{E[X \mid \mathcal{F}_t]}{E[(X)^- \mid \mathcal{F}_t]}, & \text{if } E[X|\mathcal{F}_t] > 0, \\ 0, & \text{otherwise,} \end{cases}$$

  is *not* strongly time consistent.
- An LM measure is scale invariant if $\varphi_t(\lambda X) = \varphi_t(X),\ \lambda \in \mathscr{F}_t^+$.
  No scale invariant LM measure is strongly time consistent.
  Let $A \in \mathscr{F}_s$ such that $\mathbb{P}(A) = 1/2$, and $X_n = n\mathbb{1}_A - \mathbb{1}_{A^C}, n \in \mathbb{N}$.
  By locality and scale invariance

$$\varphi_s(X_n) = \varphi_s(X_1), \quad n \in \mathbb{N}.$$

  If φ is strongly time consistent, then $\varphi_0(X_n) = \varphi_0(X_1)$. On the other hand, any reasonable performance measure should assess X_n at higher level than X_1.

Weak Time Consistency: Update Rule $\mu_t(m) = \operatorname{ess\,inf}_t m$

Proposition

Let φ be a dynamic LM-measure on L^p_T. The following are equivalent:

1) φ *is is* **weakly acceptance** *time consistent, i.e. for any* $X \in L^p_T$, $s > t$,

$$\varphi_s(X) \geq m_s \implies \varphi_t(X) \geq \operatorname{ess\,inf}_t m_s.$$

2) *For any* $X \in L^p_T$, $s > t$, $\varphi_t(X) \geq \operatorname{ess\,inf}_t \varphi_s(X)$.

3) *For any* $X \in L^p_T$, $s > t$, *with* $\mathcal{M}_t(\mathbb{P}) := \{\mathbb{Q} \in \mathcal{M}(\mathbb{P}) : \mathbb{Q}|_{\mathscr{F}_t} = \mathbb{P}|_{\mathscr{F}_t}\}$,

$$\varphi_t(X) \geq \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}_t(\mathbb{P})} E_{\mathbb{Q}}[\varphi_s(X) | \mathcal{F}_t].$$

4) *For any* $X \in L^p_T$, $s > t$, *and* $m_t \in \bar{L}^0_t$,

$$\varphi_s(X) \geq m_t \implies \varphi_t(X) \geq m_t.$$

Most of the known examples are weakly acceptance or/and rejection time consistent.

Proposition

Let φ be a dynamic LM-measure on L^p_T, and let μ be any projective update rule. If φ is μ-acceptance time consistent, then φ is weakly acceptance time consistent.

Proof. It is easy to check that $\operatorname{ess\,inf}_s X \geq \operatorname{ess\,inf}_t X$.
Then, for any $t, s \in \mathbb{T}$, $s > t$, and any $X \in L^p$, we get

$$\begin{aligned} \varphi_t(X) &\geq \mu_t(\varphi_s(X)) \geq \mu_t(\operatorname{ess\,inf}_s(\varphi_s(X))) \\ &\geq \mu_t(\operatorname{ess\,inf}_t(\varphi_s(X))) = \operatorname{ess\,inf}_t(\varphi_s(X)). \end{aligned}$$

□

Performance Based Preference Orders

Definition (Dynamic Performance Measure)

A Dynamic Acceptability Indices is a family $\{f_t\}_{t\in\mathbb{T}}$ of maps $f_t : \mathcal{X} \to \overline{L}_t(\mathscr{F}_t)$, such that, for any $X, Y \in \mathcal{X}$,

(P1) **Local:** $1_A f_t(X) = 1_A f_t(1_A X)$, for any $A \in \mathscr{F}_t$

(P2) **Monotone:** $X \leq Y \Rightarrow f_t(X) \leq f_t(Y)$

(P3) **Scale Invariant:** $f_t(\lambda X) = f_t(X)$, for any $\lambda \in L_t^p, \lambda \geq 0$.

(P4) **Quasi-Concave:** $f_t(\lambda X + (1-\lambda)Y) \geq \min\{f_t(X), f_t(Y)\}$, for any $0 \leq \lambda \leq 1, \lambda \in L_t^p$.

(P5) **Time Consistency:** For any $m_t, n_t \in L_t^0(\mathscr{F}_t)$,

$$\begin{aligned} &\text{(acceptance)} \quad f_{t+1}(X) \geq m_t \implies f_t(X) \geq m_t, \\ &\text{(rejection)} \quad f_{t+1}(X) \leq n_t \implies f_t(X) \leq n_t. \end{aligned}$$

For general theory see [BCZ14, BCC15, RGS13, BBN14]

Examples of Performance Measures

Dynamic Gain-Loss Ratio

$$\mathrm{dGLR}_t(X) = \frac{E[X \mid \mathscr{F}_t]}{E[X^- \mid \mathscr{F}_t]}.$$

dGLR is Weakly time consistent.

Sortino Ratio

$$\mathrm{SR}_t(X) = \frac{E[X \mid \mathscr{F}_t]}{\sqrt{E[(X^-)^2 \mid \mathscr{F}_t]}},$$

Risk Adjusted Return on Capital
$\mathrm{RAROC}_t(X) = E[X \mid \mathscr{F}_t]/\rho_t(X)$, where ρ is a dynamic risk measure.

Dynamic $\mathrm{TV@R}$ **Acceptability Index** - performance measure constructed using $\mathrm{TV@R}$ dynamic risk measure via dual representations.

Proposition

Let φ be a representable dynamic monetary utility measure on L^∞. The following properties are equivalent:

1) *φ is weakly acceptance time consistent.*
2) *For any $X \in L^p$ and $s, t \in \mathbb{T}$, $s > t$,*

$$\varphi_s(X) \geq 0 \Rightarrow \varphi_t(X) \geq 0.$$

3) *$\mathcal{A}_{t+1} \subseteq \mathcal{A}_t$, for any $t \in \mathbb{T}$, such that $t < T$.*
4) *For any $Q \in \mathcal{M}(P)$ and $t \in \mathbb{T}$, such that $t < T$,*

$$\alpha_t^{\min}(Q) \geq E_Q[\alpha_{t+1}^{\min}(Q) \,|\, \mathcal{F}_t],$$

where $\alpha^{\min}$ is the minimal penalty function in the robust representation of φ.

Note: $\mathrm{V@R}_\alpha$ is weakly acceptance and rejection time consistent.

Proposition

Let $\{\varphi^x\}_{x\in\mathbb{R}_+}$ *be a decreasing family of weakly acceptance/rejection time consistent dynamic LM-measures. Then,*

$$\alpha_t(X) = \sup\{x \in \mathbb{R}^+ \mid \varphi_t^x(X) \geq 0\}.$$

is a weakly acceptance/rejection time consistent dynamic LM-measure.

Let $\{\alpha_t\}_{t\in\mathbb{T}}$ *be a weakly acceptance/rejection time consistent dynamic LM-measure Then, for any* $x \in \mathbb{R}_+$*,*

$$\varphi_t^x(X) := \operatorname*{ess\,inf}_{c\in\mathbb{R}}\{c\mathbb{1}_{\{\alpha_t(X-c)\leq x\}}\},$$

is a weakly acceptance/rejection time consistent dynamic LM-measure.

Remark: Similar dual results hold true for processes, where the weak time consistency of α is replaced with semi-weak time consistency.

Time consistency wrt Robust Expectations

Proposition

Let $\mathcal{D}$ be a determining family of sets, and let φ be a dynamic LM-measure. Consider the family of maps $\phi = \{\phi_t\}_{t\in\mathbb{T}}$, $\phi_t : \bar{L}^0 \to \bar{L}^0_t$, given by the following robust expectations

$$\phi_t(m) = \operatorname*{ess\,inf}_{Z\in\mathcal{D}_t} E[Zm|\mathcal{F}_t]. \tag{6.1}$$

Then, ϕ is a projective update rule. Moreover, if φ is ϕ-acceptance time consistent, then $\{g \circ \varphi_t\}_{t\in\mathbb{T}}$ is also ϕ-acceptance time consistent, for any increasing, concave $g : \bar{\mathbb{R}} \to \mathbb{R}$.

Remark: Dynamic Coherent Risk Measures are good examples to generate update rules.

$\mathcal{D} = \{\mathcal{D}_t\}_{t\in\mathbb{T}}$ is a *determining family* if for any $t \in \mathbb{T}$, the set $\mathcal{D}_t$ satisfies the following properties: $\mathcal{D}_t \neq \varnothing$, $\mathcal{D}_t \subseteq \mathcal{P}_t$, it is L^1-closed, $\mathcal{F}_t$-convex[1], and uniformly integrable, with $\mathcal{P}_t := \{Z \in L^1 \mid Z \geq 0,\ E[Z|\mathcal{F}_t] = 1\}$.

Super(sub) martingale time consistency

For the trivial determining sets $\mathcal{D}_t = \{1\}$, the projective update rule is

$$\mu_t(m) = E[m|\mathcal{F}_t], \quad m \in \bar{L}^0.$$

Definition

Let φ be a dynamic LM-measure on L^p. We say that φ is *supermartingale (resp. submartingale) time consistent* if

$$\varphi_t(X) \geq E[\varphi_s(X)|\mathcal{F}_t], \qquad (\text{resp. } \leq)$$

for any $X \in L^p$ and $t, s \in \mathbb{T}$, $s > t$.

Any super(sub)martingale time consistent LM-measure is also weakly acceptance(rejection) time consistent.

Example

Dynamic Entropic Risk Measure with non-constant risk aversion

$$\varphi_t^{\gamma_t}(X) = \begin{cases} \frac{1}{\gamma_t} \ln E[\exp(\gamma_t X)|\mathcal{F}_t] & \text{if } \gamma_t \neq 0, \\ E[X|\mathcal{F}_t] & \text{if } \gamma_t = 0, \end{cases}$$

where $\{\gamma_t\}_{t\in\mathbb{T}}$ is such that $\gamma_t \in L_t^\infty$, $t \in \mathbb{T}$.

- $\{\varphi_t^{\gamma_t}\}_{t\in\mathbb{T}}$ is strongly time consistent if and only if $\{\gamma_t\}_{t\in\mathbb{T}}$ is a constant process [AP11];
- it is middle acceptance time consistent if and only if $\{\gamma_t\}_{t\in\mathbb{T}}$ is a non-increasing process;
- is middle rejection time consistent if and only if $\{\gamma_t\}_{t\in\mathbb{T}}$ is non-decreasing.

Other Examples: Dynamic Risk Sensitive Criterion [BCP15], and Conditional Weighted $\mathrm{V@R}$.

Time consistency induced by LM-measures

Definition

Let φ be a dynamic LM-measure on L^p. A family $\widehat{\varphi} = \{\widehat{\varphi}_t\}_{t\in\mathbb{T}}$ of maps $\widehat{\varphi}_t : \bar{L}^0 \to \bar{L}^0_t$ is **an LM-extension of φ**, if for any $t \in \mathbb{T}$, $\widehat{\varphi}_t|_{\mathcal{X}} \equiv \varphi_t$, and $\widehat{\varphi}_t$ is local and monotone on $\bar{L}_0$.

Define the collection of functions $\varphi^{\pm} = \{\varphi^{\pm}_t\}_{t\in\mathbb{T}}$, where $\varphi^{\pm}_t : \bar{L}^0 \to \bar{L}^0_t$ and

$$\varphi^+_t(X) := \operatorname*{ess\,inf}_{A\in\mathscr{F}_t}\Big[\mathbb{1}_A \operatorname*{ess\,inf}_{Y\in\mathcal{Y}^+_A(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(+\infty)\Big],$$

$$\varphi^-_t(X) := \operatorname*{ess\,sup}_{A\in\mathscr{F}_t}\Big[\mathbb{1}_A \operatorname*{ess\,sup}_{Y\in\mathcal{Y}^-_A(X)} \varphi_t(Y) + \mathbb{1}_{A^c}(-\infty)\Big],$$

where $\mathcal{Y}^+_A(X) := \{Y \in \mathcal{X} \mid \mathbb{1}_A Y \geq \mathbb{1}_A X\}$, $\mathcal{Y}^-_A(X) := \{Y \in \mathcal{X} \mid \mathbb{1}_A Y \leq \mathbb{1}_A X\}$, is called the *upper/lower LM-extension of* φ,

Proposition

Let φ be a dynamic LM-measure on L^p.

1. *The functions φ^- and φ^+ are LM-extensions of φ.*
2. *For any $\widehat{\varphi}$ LM-extension of φ*

$$\varphi_t^-(X) \leq \widehat{\varphi}_t(X) \leq \varphi_t^+(X), \quad X \in \bar{L}_0.$$

3. *$\widehat{\varphi}$ is an s-invariant update rule.*
4. *$\widehat{\varphi}$ is projective if and only if $\varphi_t(X) = X$, for $t \in \mathbb{T}$ and $X \in L^p \cap \bar{L}_t^0$.*

Definition

φ is **middle acceptance time consistent**, if it is φ^- acceptance time consistent.

For representable risk measures on L^∞, continuous from above, this is equivalent to

$$\varphi_s(X) \geq \varphi_s(Y) \implies \varphi_t(X) \geq \varphi_t(Y),$$

and, also equivalent to

$$\varphi_t(X) \geq \varphi_t(\varphi_s(X)).$$

Clearly, strong time consistent implies middle acceptance time consistent. The converse implication is not true; for counterexample see [AP11, Proposition 37]

- Similarly, φ is **middle rejection time consistency**, if it is φ^+ rejection time consistent.

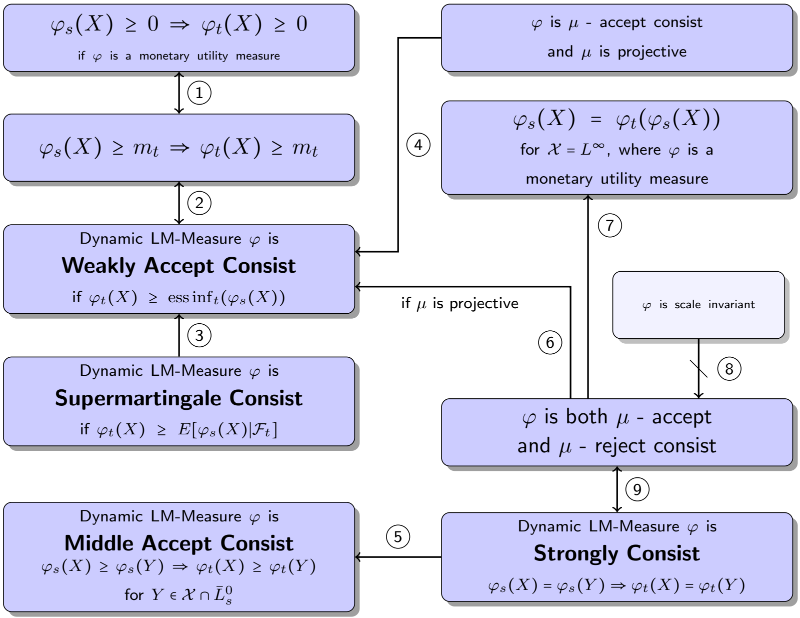
- For representable risk measures on L^∞, continuous from above, this is equivalent to

$$\varphi_s(X) \leq \varphi_s(Y) \quad \Longrightarrow \quad \varphi_t(X) \leq \varphi_t(Y),$$

or $\varphi_t(X) \geq \varphi_t(\varphi_s(X))$.

- φ is **strongly time consistent**, if there exists an LM-extension $\hat{\varphi}$, of φ, such that φ is both $\hat{\varphi}$-acceptance and $\hat{\varphi}$-rejection time consistent, or equivalently $\varphi_t(X) = \hat{\varphi}_t(\varphi_s(X)), \quad X \in \mathcal{X},\ s,t \in \mathbb{T},\ s > t.$

Figure: *Summary of results for acceptance time consistency for random variables*



Examples

		$\mathcal{X}$	WA	WR	sWA	sWR	MA	MR	STR	Sub	Sup
Cond. WV@R		L^p	$\checkmark$		$\checkmark$						$\checkmark$
TV@R AI		$\mathbb{V}^p$			$\checkmark$						
RAROC		$\mathbb{V}^p$			$\checkmark$						
dGLR		$\mathbb{V}^p$			$\checkmark$	$\checkmark$					
dEnt	$\gamma \geq 0$	L^p	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$		$\checkmark$
	$\gamma \leq 0$		$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
dEnt+	$\gamma_t \downarrow$	L^p	$\checkmark$		$\checkmark$		$\checkmark$				$\checkmark^*$
	$\gamma_t \uparrow$			$\checkmark$		$\checkmark$		$\checkmark$		$\checkmark^{**}$	
RSC		L^p	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$		

*if $\gamma_t \geq 0$, **if $\gamma_t \leq 0$

RSC = Risk Sensitive Criterion

Thank You !

The end of the talk ...
but not of the story

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