# Hörmander's condition for delayed SDEs

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# Notations

- We are interested in the following stochastic differential equation on $[0, T]$,

$$\begin{aligned} X_t &= X_0 + \int_0^t \sum_{k=0}^{m} V_k(r, X) \circ dW_r^k \\ &= X_0 + \int_0^t \sum_{k=0}^{m} V_k(X_r, X_{r-h_1}, \dots, X_{r-h_{N-1}}) \circ dW_r^k \end{aligned}$$

- for some smooth functions $V_k : (\mathbb{R}^d)^N \to \mathbb{R}^d$ and $0 = h_0 < h_1 < \cdots < h_{N-1} < T$ fixed,
- $X_t \in \mathbb{R}^d$, and $\{W_t^k\}_{t \geq 0, k=1,\dots,m}$ is a $m$-dimensional Brownian motion $W_t^0 = t$.

# Existence and uniqueness of the solutions of the SDE

**Assumption (Standing regularity assumption on $V_k$)**

*For all $k = 0, \dots, m$, the mappings $V_k : \mathbb{R}^{d\times N} \to \mathbb{R}^d$ are assumed to be smooth with bounded derivatives at all order.*

**Proposition**

*Under the standing assumptions $X_t$ exists for $t \geq 0$ and*

$$\mathbb{E}\left[\|X\|^p_{\infty,[0,t]}\right] < \infty, \text{ for all } t > 0,\ p \geq 2.$$

Proof by Picard's iteration with the norm

$$\|X\|_{\infty,[0,t]} := \sup_{s\in[0,t]} |X_s|$$

and Grönwall's inequality.

# Objectives

- Sufficient condition so that the distribution of $X_T$ is absolutely continuous with respect to the Lebesgue measure.
- Sufficient condition so that the density of $X_T$ with respect to the Lebesgue measure is smooth.
- Adapt the arguments of the Markovian case to take into account the noise coming from the delay.
- $\Longrightarrow$ Hörmander-like spanning condition of $\mathbb{R}^d$.

# Objectives

- Sufficient condition so that the distribution of $X_T$ is absolutely continuous with respect to the Lebesgue measure.
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- Adapt the arguments of the Markovian case to take into account the noise coming from the delay.
- $\Longrightarrow$ Hörmander-like spanning condition of $\mathbb{R}^d$.
- Uniformly elliptic and path-dependent case was treated by Kusuoka-Stroock(1982) and Bally-Caramellino (2016).
- Bell-Mohammed (1992), $V(X_{r-h})$ can be degenerate at $X_0$, but assumption on $X_{[-T,0]}$ to compensate it.

# Applications

- Survey of Ivanov, Kazmerchuk, Swishchuk (2003), Kuchler-Platen (2000)
- Stability
- Ergodiciy
- Numerical approximation
- Application to finance Oksendal-Sulem (2000)
- Also Arriojas, Hu, Mohammed, Pap (2007) on the delayed Black Scholes formula.

# Malliavin's derivative

- $X_T$ is Malliavin differentiable (Kusuoka-Stroock).
- Define

$$T_h := h_{N-1} \vee \sup_{i=1\dots N-1} (T - (h_i - h_{i-1})) \in (0, T).$$

- The Malliavin derivative $\mathcal{D}_t X_T$ satisfies for all $T_h \leq t \leq T$,

$$\mathcal{D}_t X_T = V(t, X) + \sum_{k=0}^{m} \int_t^T \partial_0 V_k(r, X) \mathcal{D}_t X_r \circ dW_r^k$$

where $\partial_i V_k(r, \mathbf{x}) \in \mathbb{R}^{d \times d}$ is Jacobian of $V_k$ in the $i$th component.

# Sufficient condition on the Malliavin Matrix

- Let $\mathcal{M} := \int_0^T \mathcal{D}_s X_T (\mathcal{D}_s X_T)^* ds$ be the Malliavin matrix.
- Summary of the Malliavin calculus for regularity of laws: If $\forall p > 0,\ \mathbb{E}\left(\|\mathcal{M}^{-1}\|^p\right) < \infty$ then $X_T$ has density with respect to the Lebesgue measures and its density is smooth.

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- Sufficient condition : There exist $\delta, q > 0$, $\mathcal{R} > 0$ with $\mathbb{E}[\mathcal{R}^p] < \infty$ for all $p > 1$ such that for all $\eta \in \mathbb{R}^d$ with $|\eta| = 1$ we have

$$\delta \leq \mathcal{R}\langle \eta, \mathcal{M}\eta \rangle^q.$$

- Estimate from below $\int_0^T |\eta^* \mathcal{D}_s X_T|_{\mathbb{R}^d}^2\, ds$ with $|\eta| = 1$ fixed.

# Markovian Case

- The equation becomes

$$X_t = X_0 + \int_0^t \sum_{k=0}^m V_k(r, X_r) \circ dW_r^k.$$

- If $|\eta|_{\mathbb{R}^d} = 1$, the Malliavin derivative satisfies

$$\mathcal{D}_t X_T = J_{t,T} V(t, X_t)$$

  where $J_{t,T}$ is the derivative of the flow.
- Using the evolution of $t \to J_{t,T}$ we obtain the spanning condition for smoothness of the laws of marginals.
- We define the brackets of vector fields

$$[F, G] = \partial_x G F - \partial_x F G$$

# Markovian Case

- We also define the vector spaces

$$\begin{aligned} \mathcal{V}_0 &:= \{V_k, k > 0\} \text{ and for all } j \geq 1, \\ \mathcal{V}_j &:= \mathcal{V}_{j-1} \cup \{[V_k, F] : k = 0, \ldots, m, \ F \in \mathcal{V}_{j-1}\} \\ \mathcal{V}_\infty &:= \cup_{j \geq 1} \mathcal{V}_j. \end{aligned}$$

- The correct spanning conditions is

$$span \mathcal{V}_\infty(x) = \mathbb{R}^d, \text{ for all } x \in \mathbb{R}^d.$$

- Equivalent condition

$$\inf_{|\eta|=1} \sup_{F \in \mathcal{V}_\infty} |\eta^* F(x)| > 0 \text{ for all } x \in \mathbb{R}^d$$

- Only $x = X_0$ would also be sufficient.

# Rough Paths

- Denote $\mathcal{W}_t = (W_t, W_{t-h_1}, \ldots, W_{t-h_{N-1}}) \in (\mathbb{R}^d)^N$ and we fix $\frac{1}{3} < \alpha < \frac{1}{2} < \theta < 2\alpha$.
- We need to define $\mathbb{W}^{i,j}_{s,t} := \int_s^t W_{r-h_i,s-h_i} \otimes \circ dW_{r-h_j}$. If $h_i \geq h_j$. This is the classical Stratonovich integral. For $h_i < h_j$, the anticipative integral can be deduced from the first order condition

$$\mathbb{W}^{i,j}_{s,t} + \mathbb{W}^{j,i}_{s,t} = \mathcal{W}_{s-h_i,t-h_i} \otimes \mathcal{W}_{s-h_j,t-h_j}.$$

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- Denote $\mathcal{W}_t = (W_t, W_{t-h_1}, \ldots, W_{t-h_{N-1}}) \in (\mathbb{R}^d)^N$ and we fix $\frac{1}{3} < \alpha < \frac{1}{2} < \theta < 2\alpha$.
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- One can also equivalently appeal to Nualart-Pardoux (88) and Nualart-Ocone (89) where an anticipative Stratonovich integral is defined.
- $(\mathcal{W}, \mathbb{W}) \in \mathcal{C}^\alpha$, the set of $\alpha$-Holder continuous rough paths (with iterated integrals in $\mathcal{C}^{2a}$).

# SDE as Rough integral

- We say that $(Y, Y')$ is a controlled path if

$$Y_t^{k,i} - Y_s^{k,i} = \sum_{k'=1}^{m} \sum_{i'=0}^{N-1} Y_s^{k,i,k',i'} \mathcal{W}_{s,t}^{k',i'} + O(|t-s|^{2\alpha}).$$

- We define the rough integral of a controlled rough path $Y$

$$\sum_{k,i} \int Y_r^{k,i} d\mathcal{W}_r^{k,i} := \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} Y_u \mathcal{W}_{u,v} + Y_u' \mathbb{W}_{u,v}.$$

- For all $t \in [T_h, T]$,

$$X_t = X_0 + \int_0^t \sum_{k=0}^{m} V_k(r, X_r, X_{r-h_1}, \ldots, X_{r-h_{N-1}}) d\mathcal{W}_r^{k,0}$$

$$J_{t,T} = \frac{d\text{''}\, X_T\text{''}}{d\text{''}\, X_t\text{''}} = Id + \sum_{k=0}^{m} \int_t^T \partial_0 V_k(r, X) J_{t,r} d\mathcal{W}_r^{k,0}$$

# Integration with controlled rough paths

- In particular

$$\mathcal{D}_t X_T = J_{t,T} V(t,X), \quad J_{t,T} = J_{T_h,T} J^{-1}_{T_h,t} \text{ and}$$
$$J^{-1}_{T_h,t} = Id - \sum_{k=0}^{m} \int_{T_h}^{t} J^{-1}_{T_h,r} \partial_0 V_k(r,X) d\mathcal{W}^{k,0}_r.$$

- Denote $Z_{V_j}(t) = \eta^* J_{t,T} V_j(t,X)$, we want lower bounds for

$$\mathcal{M} := \sum_{j=1}^{m} \int_0^T |\eta^* J_{s,T} V_j(s,X)|^2_{\mathbb{R}^d} \, ds = \sum_{j=1}^{m} \|Z_{V_j}\|^2_{L^2[0,T]}$$
$$\geq \sum_{j=1}^{m} \|Z_{V_j}\|^2_{L^2[T_h,T]}$$

# First estimate

- Interpolation inequality

$$\sup_{s\in[T_h,T]} \left|Z_{V_j}(s)\right| \le C_{h,T} \|Z_{V_j}\|_{L^2([T_h,T])}^{\frac{2\alpha}{2\alpha+1}} \|Z_{V_j}\|_{\alpha,[T_h,T]}^{\frac{1}{2\alpha+1}}$$

$$\le C_{h,T} \|Z_{V_j}\|_{L^2([0,T])}^{\frac{2\alpha}{2\alpha+1}} \|Z_{V_j}\|_{\alpha,[T_h,T]}^{\frac{1}{2\alpha+1}} .$$

- A first estimate

$$\sup_{j=1\ldots m} \|Z_{V_j}\|_{\infty,[T_h,T]} \le C_{h,T} \mathcal{M}^{\frac{2\alpha}{2\alpha+1}} \sum_{j=1}^{m} \|Z_{V_j}\|_{\alpha,[T_h,T]}^{\frac{1}{2\alpha+1}}.$$

- $L^\infty([T_h,T])$ estimates for $Z_{V_j}$.

# Brackets

- By Ito's formula, for all $j = 1, \ldots, m$ and $s, t \in [T_h, T]$

$$V_j(t,X) - V_j(s,X) = \sum_{k=0}^{m} \int_s^t \partial_0 V_j(r,X) V_k(r,X) d\mathcal{W}_r^{k,0} + \sum_{\substack{0 \le k \le m \\ 1 \le i \le N-1}} \int_s^t \partial_i V_j(r,X) V_k(r-h_i,X) d\mathcal{W}_r^{k,i}.$$

- To define the last term one needs to extends the $\mathcal{W}$ to take into account $t - (h_i + h_j)$ terms in the control of the integrands.
- We define the Bracket

$$[V_j, V_k] := \partial_0 V_j V_k - \partial_0 V_k V_j.$$

# Evolution of $Z_{V_j}$

- For all $s, t \in [T_h, T]$

$$Z_{V_j}(t) - Z_{V_j}(s) = \sum_{k=1}^{m} \int_s^t Z_{[V_j, V_k]}(r) d\mathcal{W}_r^{k,0}$$

$$+ \sum_{k=1}^{m} \sum_{i=1}^{N} \int_s^t Z_{\partial_i V_j(\cdot, X) V_k(\cdot - h_i, X)}(r) d\mathcal{W}_r^{k,i}$$

$$+ \int_s^t Z_{\{[V_j, V_0] + \sum_{i=1}^{N-1} \partial_i V_j(\cdot, X) V_0(\cdot - h_i, X)\}}(r) dr.$$

- $\{d\mathcal{W}_r^{k,i} : r \in (T_h, T), k = 1 \ldots m, i = 0, \ldots, N-1\}$ are the increments of a $m \times N$-dimensional Brownian Motion.

# $\theta$-Holder Roughness

## Definition

$W : [0,T] \to \mathbb{R}^m$ is $\theta$-Holder rough : there exists $L > 0$ such that for any $\varphi \in \mathbb{R}^m$, $s \in [0,T]$ and $\varepsilon \in (0,1)$, there exists $t \in [0,T]$ such that

$$|t-s| \leq \varepsilon, \text{ and } |\varphi W_{s,t}| \geq L\varepsilon^{\theta}|\varphi|.$$

## Proposition (Norris' Lemma)

*There exists a deterministic constant $C_T > 0$ and $p, q > 0$ s.t. if*

$$Z_t = \int_0^t A_s dW_s + \int_0^t B_s ds, \text{ with } A, B \text{ controlled rough paths}$$

*then* $\|A\|_{\infty,[0,T]} + \|B\|_{\infty,[0,T]} \leq C_T \left(\frac{\mathcal{R}}{L_\theta}\right)^p \|Z\|^q_{\infty,[0,T]}$.

# A variant of Norris Lemma

**Proposition**

*The path $\mathcal{W} : [T_h, T] \to \mathbb{R}^{m \times N}$ is $\theta$-Holder rough.*

**Proposition (Variant of Norris' Lemma)**

*There exists a deterministic constant $C_T > 0$ and $p, q > 0$ such that*

$$\|Z_{[V_j,V_k]}\|_{\infty,[T'_h,T]} + \|Z_{\partial_i V_j(\cdot,X) V_k(\cdot - h_i,X)}\|_{\infty,[T'_h,T]} + \|Z_{\{[V_j,V_0]+\sum_{i=1}^{N-1} \partial_i V_j(\cdot,X) V_0(\cdot - h_i,X)\}}\|_{\infty,[T'_h,T]} \leq C_{h,T} \mathcal{M}^{q_1} \left(\frac{\mathcal{R}}{L_\theta}\right)^{p_1}.$$

# A variant of Norris Lemma

## Proposition

*The path* $\mathcal{W} : [T_h, T] \to \mathbb{R}^{m\times N}$ *is* $\theta$*-Holder rough.*

## Proposition (Variant of Norris' Lemma)

*There exists a deterministic constant* $C_T > 0$ *and* $p, q > 0$ *such that*

$$\|Z_{[V_j,V_k]}\|_{\infty,[T'_h,T]} + \|Z_{\partial_i V_j(\cdot,X)V_k(\cdot-h_i,X)}\|_{\infty,[T'_h,T]}$$
$$+ \|Z_{\{[V_j,V_0]+\sum_{i=1}^{N-1}\partial_i V_j(\cdot,X)V_0(\cdot-h_i,X)\}}\|_{\infty,[T'_h,T]}$$
$$\leq C_{h,T}\mathcal{M}^{q_1}\left(\frac{\mathcal{R}}{L_\theta}\right)^{p_1}.$$

- Only $Z_{[V_j,V_k]}$ is controlled by $\mathcal{W}$.

# Link between Hörmander condition and the Malliavin derivatives

## Definition

We define the family of hyperplanes generated by the vector fields,

$$\mathcal{V}_0 := \{(s,\mathbf{x}) \to V_k(s,\mathbf{x}) : k = 1,\cdots,m\} \text{ and}$$

$$\mathcal{V}_{j+1} := \mathcal{V}_j \cup \{[F,V_k] : F \in \mathcal{V}_j, k = 1,\cdots,m\} \text{ and}$$

We also define the extension of the hyperplanes

$$\overline{\mathcal{V}}_{j+1} := \mathcal{V}_{j+1} \bigcup_{F\in\mathcal{V}_j} \{(s,\mathbf{x}) \to [F,V_0](s,\mathbf{x}) + \sum_{i=1}^{N-1} \partial_i F(s,\mathbf{x}) V_0(s-h_i,\mathbf{x})\}$$

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# Hormander

- $V_{j+1}$ is smaller than its Markovian counter part.
- $\overline{V}_{j+1}$ includes the Brackets with $V_0$.
- Additional noise coming from the delay.

**Assumption (Hörmander Condition for Delayed equations)**

*There exists $j_0$ such that $\overline{V}_{j_0}$ spans $\mathbb{R}^d$ uniformly*

$$\inf_{\mathbf{x}\in C([0,T])} \inf_{|\eta|=1} \sup_{F\in\overline{V}_{j_0}} |\eta^* F(T,\mathbf{x})| > 0$$

- Spanning uniform in $\mathbf{x}$ and in the rank $j_0$.

# Main theorem

**Theorem (Main theorem)**

*Under the standing regularity assumption of $\{V_k\}$ and the Hörmander condition $X_T$ has a smooth density with respect to the Lebesgue measure.*

- By induction one can show that

$$
\begin{aligned}
0 < \delta := \inf_{\mathbf{x} \in C([0,T])} \inf_{|\eta|=1} \sup_{F \in \overline{V}_{j_0}} |\eta^* F(T, \mathbf{x})| \\
\leq \inf_{|\eta|=1} \sup_{F \in \overline{V}_{j_0}} |\eta^* F(T, X)| \\
\leq C_{j_0,h,T} \frac{\mathcal{R}^{p_{j_0}}}{L^{p_{j_0}}} \mathcal{M}^{q_{j_0}}.
\end{aligned}
$$

# Unifomly non-degenerate

- Assume that there exists $\delta > 0$ such that for all $|\eta| = 1$

$$\sum_{k=1}^{m} |\eta^* V_k|^2 \geq \delta$$

  Then Hörmander's condition for delayed SDEs is satisfied.
- This case was treated by Kusuoka-Stroock(1982) and Bally-Caramellino (2016) without the assumption on the delay.

# Langevin Equation with Delay

Consider the diffusion in $\mathbb{R}^2$,

$$\begin{aligned} dp_t &= V_0(p_t, q_t)dt + V_1(p_t, q_t, p_{t-h}, q_{t-h}) \circ dW_t \\ dq_t &= p_t dt. \end{aligned}$$

with $V_1$ uniformly elliptic. We check the spanning condition

$$\mathcal{V}_0 = \left\{ \begin{pmatrix} V_1 \\ 0 \end{pmatrix} \right\}$$

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with $V_1$ uniformly elliptic. We check the spanning condition

$$\mathcal{V}_0 = \left\{ \begin{pmatrix} V_1 \\ 0 \end{pmatrix} \right\} = \mathcal{V}_j$$

We compute $\overline{\mathcal{V}}_1$

$$\left[ \begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \begin{pmatrix} V_0 \\ p_s \end{pmatrix} \right] + \partial_1 \begin{pmatrix} V_1 \\ 0 \end{pmatrix} \begin{pmatrix} V_0(p_{s-h}, q_{s-h}) \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ -V_1 \end{pmatrix}$$

$\Longrightarrow$ uniform spanning

# Noise from the delay

We now consider the following diffusion

$$\begin{pmatrix} p_t \\ q_t \\ r_t \end{pmatrix} = \int_0^t \begin{pmatrix} 1 \\ 1 \\ -r_{s-h} \end{pmatrix} dW_s^1 + \int_0^t \begin{pmatrix} -p_{s-h} \\ q_{s-h} \\ \sqrt{1+r_{s-h}^2} \end{pmatrix} dW_s^2$$

We check the spanning condition

$$\mathcal{V}_0 = \left\{ \begin{pmatrix} 1 \\ 1 \\ -r_{s-h} \end{pmatrix}, \begin{pmatrix} -p_{s-h} \\ q_{s-h} \\ \sqrt{1+r_{s-h}^2} \end{pmatrix} \right\}$$

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We compute the semi-brackets $\partial_1 V_2(t) V_1(t-h)$, $\partial_1 V_1(t) V_2(t-h)$ hence the subset of $\overline{\mathcal{V}}_0$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -r_{s-h} \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -\frac{r_{s-h} r_{s-2h}}{\sqrt{1+r_{s-h}^2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\sqrt{1+r_{s-2h}^2} \end{pmatrix} \right\} \subset \overline{\mathcal{V}}_0$$

$\implies$ uniform spanning

THANK YOU!