Mean Field Games: Basic theory and generalizations

Minyi Huang

School of Mathematics and Statistics
Carleton University
Ottawa, Canada

University of Southern California, Los Angeles, Mar 2018

Outline of talk

- Mean field games (MFGs)
  - Historical background of noncooperative games
  - Simple examples of MFG; nonlinear diffusion dynamics: "interacting particle" modeling
  - Fundamental approaches
  - Generalizations
- Three concrete MFG models
  - Competitive binary Markov decision process (MDP)
  - Stochastic growth model
  - LQ mean field social optimization with a major player
- References

A mean field game is a situation of stochastic (dynamic) decision making where

- each agent interacts with the **aggregate effect** of all other agents;
- agents are **non-cooperative**.

Example: Hybrid electric vehicle recharging control

interaction through **price**

Applications of MFG:

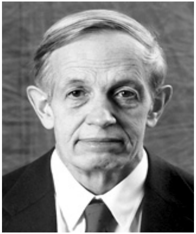
- Economic theory
- Finance
- Communication networks
- Social opinion formation
- Power systems
- Electric vehicle recharging control
- Public health (vaccination games)
- Cyber-security problems (botnets)
- ......

Historical Background of Noncooperative Games

1. Cournot duopoly   Cournot equilibrium, 1838
2. 2 person zero-sum   von Neumann's minimax theorem, 1928
3. N-person nonzero-sum   Nash equilibrium, 1950


Antoine Cournot
(1801-1877)


John von Neumann
(1903-1957)


John Nash
(1928-2015)

Two directions for generalizations:

- Include **dynamics**
  - L. Shapley (1953), MDP model; Rufus Isaacs (1965), *Differential games*, Wiley, 1965 (work of 1950s).
- Consider **large populations** of players

Technical challenges arise ...

- the **curse of dimensionality** when the number of players is large
- Example: 50 players, each having 2 states, 2 actions. Need a system configuration: $4^{50} = 1.27 \times 10^{30}$

von Neumann and Morgenstern's version on games with a large number of players –

"... When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible, and that the above difficulties may recede and a more conventional theory becomes possible."

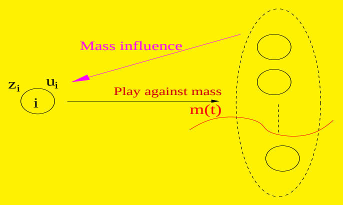
"... In all fairness to the traditional point of view this much ought to be said: It is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size. An almost exact theory of a gas, containing about 10^{25} freely moving particles, is incomparably easier than that of the solar system, made up of 9 major bodies ... This is, of course, due to the excellent possibility of applying the laws of statistics and probability in the first case."

J. von Neumann and O. Morgenstern (1944), *Theory of games and economic behavior*, Princeton University Press, p. 12.

Early efforts to deal with large populations

- J. W. Milnor and L.S. Shapley (1978); R. J. Aumann (1975). This is for cooperative games
- E. J. Green (1984). Non-cooperative games
- M. Ali Khan and Y. Sun (2002). Survey on "Non-cooperative games with many players". Static models.
- B. Jovanovic and R. W. Rosenthal (1988). Anonymous sequential games. It considers distributional strategies for an infinite population. However, individual behaviour is not addressed.
- ...

Modeling of mean field games



- Model: Each player interacts with the empirical state distribution $\delta_z^{(N)}$ of N players via dynamics and/or costs (see examples)
  - Individual state and control (z_i, u_i), $\quad \delta_{z(t)}^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{z_i(t)}$
- Objective: Overcome dimensionality difficulty

MFG – Example 1: **Continuous time LQ**

$$dX_t^i = (A_{\theta_i} X_t^i + Bu_t^i + GX_t^{(N)})dt + DdW_t^i, \quad 1 \le i \le N$$
$$J_i(u^i, u^{-i}) = E\int_0^\infty e^{-\rho t}\left\{|X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i\right\} dt$$

To relate to the empirical distribution, denote

$$\delta_{X(t)}^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta_{X^i(t)}$$
$$X_t^{(N)} = \frac{1}{N}\sum_{i=1}^N X_t^i = \int_{\mathbb{R}} y\delta_{X(t)}^{(N)}(dy)$$

MFG – Example 2: **Discrete time Markov decision processes (MDPs)**

- N MDPs

$$(x_t^i, a_t^i), \quad 1 \le i \le N,$$

where state x_t^i has transitions affected by action a_t^i but not by $(a_t^1, \ldots, a_t^{i-1}, a_t^{i+1}, \ldots, a_t^N)$.

- Player i has cost

$$J_i = E \sum_{t=0}^{T} \rho^t l(x_t^i, x_t^{(N)}, a_t^i), \qquad \rho \in (0, 1]$$

MFG – Example 3: **Diffusion model** of a Nash game of N players:

$$dx_i = \frac{1}{N}\sum_{j=1}^{N} f(x_i, u_i, x_j)dt + \sigma dw_i, \quad 1 \le i \le N, \quad t \ge 0,$$

$$J_i(u_i, u_{-i}) = E\int_0^T \Big[\frac{1}{N}\sum_{j=1}^{N} L(x_i, u_i, x_j)\Big]dt, \quad T < \infty.$$

- Mean field coupling in dynamics and costs
- Denote $\delta_{x(t)}^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}$. Then

$$\frac{1}{N}\sum_{j=1}^{N} f(x_i, u_i, x_j) = \int_{\mathbb{R}^n} f(x_i, u_i, y)\delta_{x(t)}^{(N)}(dy)$$

The traditional approach: infeasibility

- Rewrite vector mean field dynamics (controlled diffusion):

$$dx(t) = f_N(x(t), u_1(t), \dots, u_N(t))dt + \sigma_N dW(t).$$

- Cost of agent $i \in \{1, \dots, N\}$: $J_i(u_i, u_{-i}) = E \int_0^T l_i(x(t), u_i, u_{-i})dt$ where u_{-i} is the set of controls of all other agents
- Dynamic programming (N coupled HJB equations):

$$\begin{cases} 0 = \frac{\partial v_i}{\partial t} + \min_{u_i} \left[f_N^T \frac{\partial v_i}{\partial x} + \frac{1}{2} \mathrm{Tr}\left(\frac{\partial^2 v_i}{\partial x^2} \sigma_N \sigma_N^T \right) + l_i \right], \\ v_i(T, x) = 0, \quad 1 \le i \le N \end{cases}$$

  - Need too much information since the HJBs give an individual strategy of the form $u_i(t, x_1, \dots, x_N)$.
  - Computation is heavy, or impossible in nonlinear systems.
  - Need a new methodology: mean field stochastic control theory!

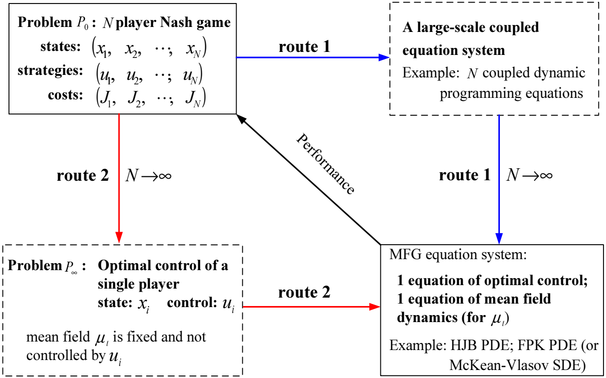


Figure : The fundamental diagram of MFG theory

- The red route: **top-down** (impose consistency in the mean field approximations). See Huang, Malhame, and Caines (2006; nonlinear case), Huang, Caines, and Malhame (2003, 2007; LQ case)
- The blue route: **bottom-up**. See Lasry and Lions (2006, 2007; nonlinear case, restricted to decentralized information)

Solution of Example 1: Problem P_0 –

$$dX_t^i = (A_{\theta_i} X_t^i + Bu_t^i + GX_t^{(N)})dt + DdW_t^i, \quad 1 \le i \le N$$
$$J_i(u^i, u^{-i}) = E\int_0^\infty e^{-\rho t}\left\{|X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i\right\} dt$$

Riccati equation: $\rho\Pi = \Pi A + A^T\Pi - \Pi BR^{-1}B^T\Pi + Q$

The MFG solution equation system:

$$\begin{cases} \frac{d\bar{X}_t}{dt} = (A - BR^{-1}B^T\Pi + G)\bar{X}_t - BR^{-1}B^T s_t, & \text{I.C.} \quad \bar{X}_0 \\ \rho s_t = \frac{ds_t}{dt} + (A^T - \Pi BR^{-1}B^T)s_t + \Pi G\bar{X}_t - Q(\Gamma\bar{X}_t + \eta) \end{cases}$$

The set of **decentralized** strategies for the N players:

$$\hat{u}_t^i = -R^{-1}B^T(\Pi X_t^i + s_t), \quad 1 \le i \le N.$$

Results:

i) **Existence** for ODE via fixed point argument.
ii) The set of strategies is an ϵ-**Nash equilibrium**

The fundamental diagram with diffusion dynamics:

$$\begin{cases} P_0\text{—Game with } N \text{ players; Example} \\ dx_i = f(x_i, u_i, \delta_x^{(N)})dt + \sigma(\cdots)dw_i \\ J_i(u_i, u_{-i}) = E\int_0^T l(x_i, u_i, \delta_x^{(N)})dt \\ \delta_x^{(N)} : \text{empirical distribution of } (x_j)_{j=1}^N \end{cases} \quad \xrightarrow{\text{solution}} \quad \begin{array}{l} \text{HJBs coupled via densities } p_{i,t}^N,\ 1 \le i \le N \\ +N \text{ Fokker-Planck-Kolmogorov equations} \\ u_i \text{ adapted to } \sigma(w_i(s), s \le t) \\ (i.e., \text{ restrict to decentralized info} \\ \text{for } N \text{ players}); \text{ so giving } u_i^N(t, x_i) \end{array}$$

$\downarrow$construct $\qquad \nwarrow$performance? $\qquad$ (subseq. convergence)$\downarrow N \to \infty$

$$\begin{cases} P_\infty\text{—Limiting problem, 1 player} \\ dx_i = f(x_i, u_i, \mu_t)dt + \sigma(\cdots)dw_i \\ \bar{J}_i(u_i) = E\int_0^T l(x_i, u_i, \mu_t)dt \\ \text{Freeze } \mu_t, \text{ as approx. of } \delta_x^{(N)} \end{cases} \quad \xrightarrow{\text{solution}} \quad \begin{cases} \hat{u}_i(t, x_i) : \text{ optimal response} \\ \text{HJB (with } v(T, \cdot) \text{ given)} : \\ -v_t = \inf_{u_i}(f^T v_{x_i} + l + \frac{1}{2}Tr[\sigma\sigma^T v_{x_i x_i}]) \\ \text{Fokker-Planck-Kolmogorov} : \\ p_t = -div(fp) + \sum((\frac{\sigma\sigma^T}{2})_{jk}p)_{x_i^j x_i^k} \\ \text{Coupled via } \mu_t \text{ (w. density } p_t; p_0 \text{ given)} \end{cases}$$

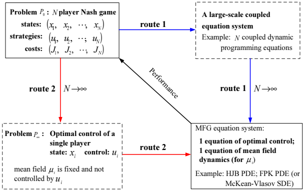
- The consistency based approach (red) is more popular; related to ideas in statistical physics (McKean-Vlasov eqn); FPK can be replaced by an MV-SDE

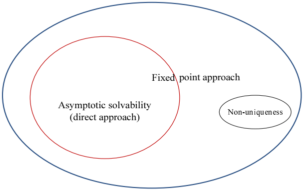
A brief note:

Due to time constraint, this talk did not cover the so-called mean field type control (with a single decision maker)

One such example is mean-variance portfolio optimization, where the objective contains $Var(X(t))$ which depends on the mean in a nonlinear manner.

Comparison of the two approaches in an LQ setting (Huang and Zhou'18):
blue route (direct approach); red route (fixed point approach)





$$dX_t^i = (A_{\theta_i} X_t^i + Bu_t^i + G X_t^{(N)})dt + DdW_t^i, \quad 1 \le i \le N$$

$$J_i(u^i, u^{-i}) = E \int_0^T \left\{ |X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i \right\} dt$$

Definition: Asymptotic solvability – The N coupled dynamic programming equations have solutions in addition to mild regularity requirement on the solution behavior. Its necessary and sufficient condition: A non-symmetric Riccati ODE in $\mathbb{R}^n$ has a solution on $[0, T]$ where $X_t^i \in \mathbb{R}^n$.

- Major players with strong influence (example: institutional trader and many smaller traders)
- Robustness
- Cooperative decision(team)
- Partial information
- ...

(I) Competitive MDP modeling of MFGs

- Except for LQ cases, general nonlinear MFGs rarely have closed-form solutions
- We introduce this class of MDP models which have relatively simple solutions (threshold policy)

The MF MDP model: (Huang and Ma, 2016)

- N players (or agents $\mathcal{A}_i$, $1 \le i \le N$) with states x_t^i, $t \in \mathbb{Z}_+$, as controlled Markov processes (no coupling)
  - State space: $\mathbf{S} = [0, 1]$
  - Action space: $\mathbf{A} = \{a_0, a_1\}$. a_0: inaction
- Interpret state as unfitness or risk
- Agents are **coupled** via the costs

Examples of binary action space (action or inaction)

- maintenance of equipment;
- network security games;
- vaccination games, etc.

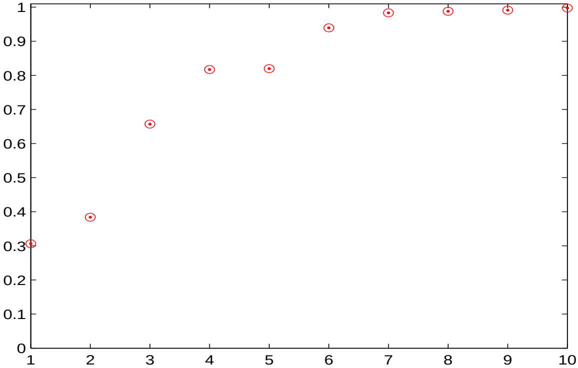
Dynamics:

The controlled transition kernel for x_t^i. For $t \geq 0$ and $x \in \mathbf{S}$,

$$P(x_{t+1}^i \in B | x_t^i = x, a_t^i = a_0) = Q_0(B|x),$$
$$P(x_{t+1}^i = 0 | x_t^i = x, a_t^i = a_1) = 1,$$

- $Q_0(\cdot|x)$: **stochastic kernel** defined for $B \in \mathcal{B}(\mathbf{S})$ (Borel sets).
- $Q_0([x, 1]|x) = 1$. Recall $a_0 =$ inaction.
  So the state **gets worse** under inaction.
- Transition of x_t^i is not affected by other a_t^j, $j \neq i$.

The stochastic deterioration is similar to hazard rate modelling in the maintenance literature (Bensoussan and Sehti, 2007; Grall et al. 2002)



x_t^i under inaction. It is getting worse and worse.

The cost of $\mathcal{A}_i$:

$$J_i = E \sum_{t=0}^{T} \rho^t c(x_t^i, x_t^{(N)}, a_t^i), \quad 1 \le i \le N.$$

- $0 < T \le \infty$; $\rho \in (0,1)$: discount factor.
- Population average state: $x_t^{(N)} = \frac{1}{N} \sum_{i=1}^N x_t^i$.
- The one stage cost:

$$c(x_t^i, x_t^{(N)}, a_t^i) = R(x_t^i, x_t^{(N)}) + \gamma 1_{\{a_t^i = a_1\}}$$

  Motivation: network maintenance game
- $R \ge 0$: unfitness-related cost; $\gamma > 0$: the effort cost

Stationary equation system for MFG

$$V(x) = \min \left[\rho \int_0^1 V(y) Q_0(dy|x) + R(x,z), \quad \rho V(0) + R(x,z) + \gamma \right]$$

$$z = \int_0^1 x \pi(dx)$$

where π is a probability measure on $[0,1]$

Remark: The second equation means $z = Ex_t^i$ in steady state

Existence and Uniqueness can be developed under technical assumptions.

Solution property: threshold policy

Assumptions:

(A1) $\{x_0^i, i \geq 1\}$ are independent random variables taking values in $\mathbf{S}$.

(A2) $R(x,z)$ is a continuous function on $\mathbf{S} \times \mathbf{S}$. For each fixed z, $R(\cdot, z)$ is strictly increasing.

(A3) i) $Q_0(\cdot|x)$ satisfies $Q_0([x,1]|x) = 1$ for any x, and is strictly stochastically increasing; ii) $Q_0(\cdot|x)$ has a positive density for all $x < 1$.

(A4) $R(x, \cdot)$ is increasing for each fixed x.

(A5) $\gamma > \beta \max_z \int_0^1 [R(y,z) - R(0,z)] Q_0(dy|0)$.

Remarks:

- Montonicity in (A2): cost increases when state is poorer.
- (A3)-i) means dominance of distributions
- (A5) Effort cost should not be too low; this prevents zero action threshold

Theorem (existence) Assume (A1)-(A5). Then

$$V(x) = \min \left[\rho \int_0^1 V(y) Q_0(dy|x) + R(x,z), \quad \rho V(0) + R(x,z) + \gamma\right]$$

$$z = \int_0^1 x\pi(dx)$$

has **a solution** (V, z, π, a^i) for which the best response a^i is a threshold policy. Uniqueness holds if we further assume $R(x,z) = R_1(x)R_2(z)$ and $R_2 > 0$ is strictly increasing on $\mathbf{S}$.

- a^i: $x \in [0,1] \mapsto \{a_0, a_1\}$, is implicitly specified by the first equation as a threshold policy
- Show each θ-threshold policy leads to a limiting distribution π_θ (verify Doebline's condition for each $\theta \in (0,1)$); in fact $\inf_{x\in\mathbf{S}} P(x_4 = 0|x_0 = x) \geq \eta > 0$.

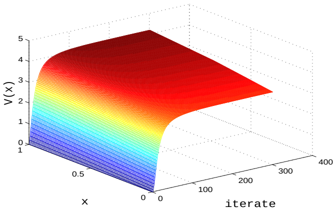


Figure : $V(x)$ defined on $[0, 1]$

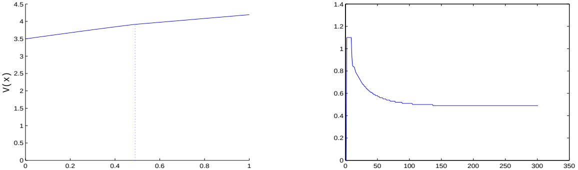


Figure : Left: $V(x)$ (threshold 0.49) Right: search of the threshold 0.49

(II) – Stochastic growth

The idea of relative performance in economic literature:

- Abel (1990), Amer. Econ. Rev.
- Hori and Shibata (2010), J. Optim. Theory Appl.
- Turnovsky and Monterio (2007), Euro. Econ. Rev.
- . . .

MFG with relative performance (Espinosa and Touzi, 2013)

- GBM dynamics for risky assets
- The performance of agent (manager) i ($i = 1, 2, \ldots, N$):

$$EU\left[(1-\lambda)X_T^i + \lambda(X_T^i - X_T^{(-i)})\right], \qquad X_T^{(-i)} = \frac{1}{N-1}\sum_{j\neq i} X_T^j,\ 0 < \lambda < 1$$

Feature: This is difference-like comparison

Our growth model (Huang and Nguyen, AMO'16):

- Cobb-Douglas production function
- Relative utility based performance

Dynamics of the N agents:

$$dX_t^i = \left[A(X_t^i)^\alpha - \delta X_t^i - C_t^i\right] dt - \sigma X_t^i dW_t^i, \quad 1 \le i \le N,$$

- X_t^i: capital stock, $X_0^i > 0$, $EX_0^i < \infty$, C_t^i: consumption rate
- Ax^α, $\alpha \in (0,1)$: Cobb-Douglas production function, $0 < \alpha < 1$, $A > 0$
- $\delta dt + \sigma dW_t^i$: stochastic depreciation (see e.g. Wälde'11, Feicht and Stummer'10 for stochastic depreciation modeling)
- $\{W_t^i, 1 \le i \le N\}$ are i.i.d. standard Brownian motions. The i.i.d. initial states $\{X_0^i, 1 \le i \le N\}$ are also independent of the N Brownian motions.

The utility functional of agent i:

$$J_i(C^1, \dots, C^N) = E\left[\int_0^T e^{-\rho t} U(C_t^i, C_t^{(N,\gamma)}) dt + e^{-\rho T} S(X_T)\right],$$

where $C_t^{(N,\gamma)} = \frac{1}{N}\sum_{i=1}^N (C_t^i)^\gamma$, $\gamma \in (0,1)$. Take $S(x) = \frac{\eta x^\gamma}{\gamma}$, $\eta > 0$.

Take the utility function

$$U(C_t^i, C_t^{(N,\gamma)}) = \frac{1}{\gamma}(C_t^i)^{\gamma(1-\lambda)}\left[\frac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}}\right]^\lambda$$
$$\left(= \left[\tfrac{1}{\gamma}(C_t^i)^\gamma\right]^{1-\lambda}\left[\tfrac{1}{\gamma}\tfrac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}}\right]^\lambda =: U_0^{1-\lambda} U_1^\lambda\right),$$

as **weighted geometric mean** of U_0 (own utility), U_1 (relative utility).

Further take $\gamma = 1 - \alpha$, i.e., equalizing the coefficient of the relative risk aversion to capital share; useful for closed-form solution

Results:

- The individual strategy is a linear feedback

$$\hat{C}_t^i = b_t \Big[e^{a(t-T)} \eta^{\frac{1}{1-\gamma}} + e^{at} \int_t^T e^{-as} b_s ds \Big]^{-1} X_t^i, \qquad 1 \le i \le N.$$

a: constant; b: determined from fixed point equation (FPE)

$$b_t = \Gamma(b)_t, \quad t \in [0, T].$$

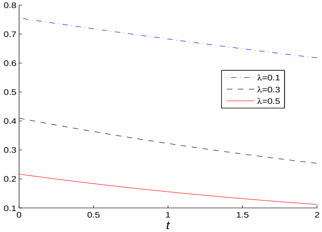
- Existence of a solution to the FPE by a contraction argument
- The set of strategies is an ε-Nash equilibrium.

We solve the fixed equation $b = \Gamma(b)$ with the following parameters

$$T = 2,\ A = 1,\ \delta = 0.05,\ \gamma = 0.6,\ \eta = 0.2,\ \rho = 0.04,\ \sigma = 0.08,$$

- λ will take three different values $0.1, 0.3, 0.5$ for comparisons.
- See Feicht and Stummer (2010) for typical parameter values in capital growth models with stochastic depreciation.
- Time is discretized with step size 0.01.

Numerical example



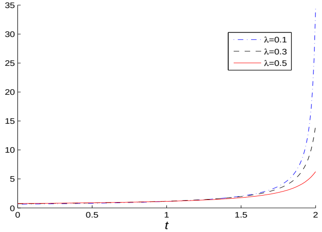


Figure : *Left*: b_t solved from $b = \Gamma(b)$; *right*: $b_t \Gamma_0(b)_t$ (as control gain)

When the agent is more concerned with the relative utility (i.e., larger λ), it consumes with more caution during the late stage.

Numerical example

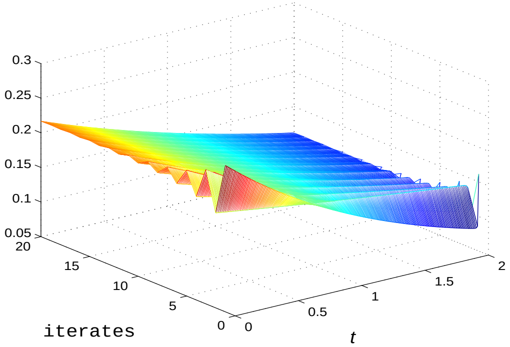


Figure : The computation of b_t in the first 20 iterates by operator Γ, $\lambda = 0.5$.

(III) – Mean field teams with a major player

Related work on social optimization

- Huang, Caines and Malhamé (TAC, 2012) – LQ mean field team with peers
- Sen, Huang and Malhamé (CDC'16) – Nonlinear diffusions with peers
- Huang and Nguyen (IFAC'2011, CDC'16) – LQ model with a major player, no dynamic coupling

Motivation for cooperative decision (mean field team)

- Manage space heaters in large buildings (hotel, apartment building, etc); they can run cooperatively to maintain comfort and good average load (as a mean field)
- Kizilkale and Malhame (2016) considered related collective target tracking reflecting partial cooperation; linear SDE temperature dynamics

Dynamics of the major player $\mathcal{A}_0$, and N minor players $\mathcal{A}_i$:

$$
\begin{aligned}
dx_{0,t}^N &= (A_0 x_{0,t}^N + B_0 u_{0,t}^N + F_0 x_t^{(N)})dt + D_0 dW_{0,t}, \\
dx_{i,t}^N &= (A x_{i,t}^N + B u_{i,t}^N + F x_t^{(N)} + G x_{0,t}^N)dt + D dW_{i,t}, \quad 1 \le i \le N.
\end{aligned}
$$

(A1) The initial states $x_{j,0}^N = x_j(0)$ for $j \ge 0$. $\{x_j(0), 0 \le j \le N\}$ are independent, and for all $1 \le i \le N$, $Ex_i(0) = \mu_0$. $\sup_i E|x_i(0)|^2 \le c$ for a constant c independent of N.

Note: The condition of equal initial means can be generalized.

The cost for $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \le i \le N$:

$$J_0(u_0^N, u_{-0}^N) = E\int_0^T \{|x_0^N - \Phi(x^{(N)})|_{Q_0}^2 + (u_0^N)^T R_0 u_0^N\} dt,$$

$$J_i(u_i^N, u_{-i}^N) = E\int_0^T \{|x_i^N - \Psi(x_0^N, x^{(N)})|_{Q}^2 + (u_i^N)^T R u_i^N\} dt,$$

where $Q_0 \ge 0$, $Q \ge 0$ and $R_0 > 0$, $R > 0$,

- $u_{-j}^N = (u_0^N, \ldots, u_{j-1}^N, u_{j+1}^N, \ldots, u_N^N)$, $\quad x^{(N)} = \frac{1}{N}\sum_{i=1}^N x_i^N$,
- $\Phi(x^{(N)}) = H_0 x^{(N)} + \eta_0$, $\quad \Psi(x_0^N, x^{(N)}) = H x_0^N + \hat{H} x^{(N)} + \eta$.

The **social cost**:

$$J_{\mathrm{soc}}^{(N)}(u^N) = J_0 + \frac{\lambda}{N}\sum_{k=1}^N J_k,$$

where $u^N = (u_0^N, u_1^N, \ldots, u_N^N)$ and $\lambda > 0$.

Notation:

- $u^N = (u_0^N, u_1^N, \cdots, u_N^N)$
- $u_{-j}^N = (u_0^N, \ldots, u_{j-1}^N, u_{j+1}^N, \ldots, u_N^N)$, $j \geq 0$
- $x^{(N)} = \frac{1}{N}\sum_{i=1}^N x_i^N$, $\hat{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{x}_i^N$, etc
- $\hat{u}^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{u}_i^N$
- $\hat{x}_{-i}^{(N)} = \frac{1}{N}\sum_{j\neq i}^N \hat{x}_j^N$.
- $\tilde{x}_{-i}^{(N)} = \frac{1}{N}\sum_{j\neq i}^N \tilde{x}_j^N$.
- x_0^∞, x_i^∞, u_i^∞, etc. for the limiting model
- m, $\hat{m}$, $\tilde{m}$ for the mean field

Main result 1. The solution of the MFT leads to the FBSDE

$$\begin{aligned}
d\hat{x}_0^\infty &= (A_0\hat{x}_0^\infty + B_0R_0^{-1}B_0^Tp_0 + F_0\hat{m})dt + D_0dW_0,\\
d\hat{m} &= [(A+F)\hat{m} + G\hat{x}_0^\infty + B(\lambda R)^{-1}B^Tp]dt,\\
dp_0 &= \{-A_0^Tp_0 - G^Tp + Q_0[\hat{x}_0^\infty - (H_0\hat{m}+\eta_0)]\\
&\quad - H^T\lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi_0dW_0,\\
dp &= \{-F_0^Tp_0 - (A+F)^Tp - H_0^TQ_0[\hat{x}_0^\infty - (H_0\hat{m}+\eta_0)]\\
&\quad + (I-\hat{H})^T\lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi dW_0,\\
d\hat{x}_i^\infty &= [A\hat{x}_i^\infty + B(\lambda R)^{-1}B^Tq_i + F\hat{m} + G\hat{x}_0]dt + DdW_i,\\
dq_i &= \{-F_0^Tp_0 - F^Tp - A^Tq_i - H_0^TQ_0[\hat{x}_0^\infty - (H_0\hat{m}+\eta_0)]\\
&\quad + \lambda Q[\hat{x}_i^\infty - (H\hat{x}_0^\infty + \hat{H}\hat{m} + \eta)]\\
&\quad - \hat{H}\lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta_i^a dW_0 + \zeta_i^b dW_i.
\end{aligned}$$

where $\hat{x}_{0,0}^\infty = x_{0,0}^N$, $\hat{m}_0 = \mu_0$, $\hat{x}_{i,0}^\infty = x_{i,0}^N$, $p_0(T) = p(T) = p_i(T) = 0$.

Theorem. This FBSDE has a unique solution.

Remark 1: General FBSDEs do not always have a solution.

Main result 2. Performance gap

Social Optimality Theorem We have

$$|J_{\mathrm{soc}}^{(N)}(\hat{u}) - \inf_{u} J_{\mathrm{soc}}^{(N)}(u)| = O(1/\sqrt{N}),$$

where each u_j^N, $0 \le j \le N$ within u is in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$, and

$$\hat{u}_0^N = \hat{u}_0^\infty = R_0^{-1} B_0^T p_0, \quad \hat{u}_i^N = \hat{u}_i^\infty = (\lambda R)^{-1} B^T q_i.$$

□

- We can further show that p_0 is a linear function of $(\hat{x}_0^\infty, \hat{m})$.
- We may choose $\mathcal{F}_t$ as the σ-algebra $\mathcal{F}_t^{x.(0),W.} \triangleq \sigma(x_j(0), W_j(\tau), 0 \le j \le N, \tau \le t)$.

How is the FBSDE introduced?

- If $(\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_N)$ (as progressively measurable processes) is a social optimum, a perturbation δu_k in any single component $\hat{u}_k$ will make the social cost $J_{\rm soc}^{(N)}$ worse off (this property is called **person-by-person optimality** in team decision theory).
- This provides **variational conditions**, one for the major player and the other for a representative minor player.
- Derive the mean field limit and impose consistency.

Basic references and surveys on mean field games –

Huang, Caines, and Malhame. 2003 CDC, IEEE TAC'07.
Huang, Malhame and Caines, CIS'06.
Lasry and Lions. Mean field games, 2007.
Caines, IEEE Control Systems Society Bode Lecture, 2009;
Bensoussan, Frehse, and Yam. Springer Brief 2013.
Caines. Mean field games, in Encyclopedia of Systems and Control, Springer, 2014.
Cardaliaguet. Notes on mean field games, 2012.
Gomes and Saud. Mean field games models – a brief survey. 2014.
Caines, Huang and Malhame (2017) Mean field games. (Springer handbook article)
Carmona and Delarue (2017) Mean field games, springer.

Mean field game with MDP modeling –

Huang, Malhame, and Caines, CDC'04; Weintraub, Benkard, and Van Roy (2008). Econometrica; Adlakha, Johari, and Weintraub (2015). J. Econ. Theory.

Thank you!