Optimal Portfolio under Fractional Stochastic Environment

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Mathematical Finance Colloquium
University of Southern California
Jan 29, 2018

Portfolio Optimization: Merton's Problem

An investor manages her portfolio by investing on a riskless asset B_t and one risky asset S_t (single asset for simplicity)

$$\begin{cases} \mathrm{d}B_t = rB_t\,\mathrm{d}t \\ \mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \end{cases}$$

- π_t – amount of wealth invested in the risky asset at time t
- X_t^π – the wealth process associated to the strategy π

$$\mathrm{d}X_t^\pi = \frac{\pi_t}{S_t}\,\mathrm{d}S_t + \frac{X_t^\pi - \pi_t}{B_t}\,\mathrm{d}B_t \quad \text{(self-financing)}$$
$$= (rX_t^\pi + \pi_t(\mu - r))\,\mathrm{d}t + \pi_t\sigma\,\mathrm{d}W_t$$

Objective:

$$M(t,x;\lambda) := \sup_{\pi\in\mathcal{A}(x,t)} \mathbb{E}\left[U(X_T^\pi)|X_t^\pi = x\right]$$

where $\mathcal{A}(x)$ contains all admissible π and $U(x)$ is a utility function on $\mathbb{R}^+$

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Stochastic Volatility

- In Merton's work, μ and σ are constant, complete market
- Empirical studies reveal that σ exhibits "random" variation
- Implied volatility skew or smile
- Stochastic volatility model: $\mu(Y_t), \sigma(Y_t) \to$ incomplete market
- Rough Fractional Stochastic volatility:
  - Gatheral, Jaisson and Rosenbaum '14
  - Jaisson, Rosenbaum '16
  - Omar, Masaaki and Rosenbaum '16

We study the Merton problem under slowly varying / fast mean-reverting fractional stochastic environment:

Nonlinear + Non-Markovian $\to$ HJB PDE not available

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Related Literature

- Option Pricing + Markovian modeling:
  Fouque, Papanicolaou, Sircar and Solna '11 (CUP)
- Portfolio Optimization + Markovian modeling:
  Fouque, Sircar and Zariphopoulou '13 (MF)
  Fouque and H. '16 (SICON)
- Option Pricing + Non-Markovian modeling:
  Garnier and Solna '15 (SIFIN), '16 (MF)
- Portfolio Optimization + Non-Markovian modeling:
  Fouque and H. (slow factor, under revision at MF)
  Fouque and H. (fast factor, under revision at SIFIN)

A General Non-Markovian Model

Dynamics of the risky asset S_t

$$\begin{cases} \mathrm{d}S_t = S_t\left[\mu(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}W_t\right], \\ Y_t\text{: a general stochastic process, } \mathcal{G}_t := \sigma\left\{\left(W^Y\right)_{0\leq u\leq t}\right\}\text{-adapted,} \end{cases}$$

with
$$\mathrm{d}\left\langle W, W^Y\right\rangle_t = \rho\,\mathrm{d}t.$$

Dynamics of the wealth process X_t (assume $r = 0$ for simplicity):

$$\mathrm{d}X_t^\pi = \pi_t\mu(Y_t)\,\mathrm{d}t + \pi_t\sigma(Y_t)\,\mathrm{d}W_t$$

Define the value process V_t by

$$V_t := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t} \mathbb{E}\left[U(X_T^\pi)\middle|\,\mathcal{F}_t\right]$$

where $U(x)$ is of power type $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma > 0$.

If Y_t is Markovian ...

For example, Y_t is a diffusion process

$$\mathrm{d}Y_t = k(Y_t)\,\mathrm{d}t + h(Y_t)\,\mathrm{d}W_t^Y,$$

$V_t \equiv V(t,x,y) \rightarrow$ characterized by a nonlinear HJB PDE

With a distortion transformation[1] $V(t,x,y) = \frac{x^{1-\gamma}}{1-\gamma}\Psi(t,y)^q$, Ψ solves the linear PDE

$$\Psi_t + \left(\frac{1}{2}h^2(y)\partial_{yy} + k(y)\partial_y + \rho\frac{1-\gamma}{\gamma}\lambda(y)h(y)\partial_y\right)\Psi + \frac{1-\gamma}{2q\gamma}\lambda^2(y)\Psi = 0,$$

and has the probabilistic representation

$$\Psi(t,y) = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\middle| Y_t = y\right].$$

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If Y_t is Markovian + Slowly Varying...

For example, Y_t is a diffusion process

$$\mathrm{d}Y_t = \delta k(Y_t)\,\mathrm{d}t + \sqrt{\delta}h(Y_t)\,\mathrm{d}W_t^Y,$$

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and has the expansion

$$\Psi(t,y) = \psi^{(0)}(t,y) + \sqrt{\delta}\psi^{(1)}(t,y) + \delta\psi^{(2)}(t,y) + \cdots.$$

[1] Zariphopoulou '99

In General: Martingale Distortion Transformation[2]

- The value process V_t is given by

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right)\right]^q, \quad \lambda(y) = \frac{\mu(y)}{\sigma(y)}$$

where under $\widetilde{\mathbb{P}}$, $\widetilde{W}_t^Y := W_t^Y + \int_0^t a_s\,\mathrm{d}s$ is a standard BM.

- The optimal strategy π^* is

$$\pi_t^* = \left[\frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{\rho q \xi_t}{\gamma\sigma(Y_t)}\right] X_t$$

where ξ_t is given by the martingale representation $\mathrm{d}M_t = M_t \xi_t\,\mathrm{d}\widetilde{W}_t^Y$ and M_t is

$$M_t = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_0^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right]$$

[2] Tehranchi '04: different utility function, proof and assumptions

Remarks

- only works for one factor model
- assumptions: integrability conditions of ξ_t, X_t^π and π_t
- $\gamma = 1 \to$ case of log utility, can be treated separately
- degenerate case $\lambda(y) = \lambda_0$, M_t is a constant martingale, $\xi_t = 0$

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda_0^2(T-t)}, \quad \pi_t^* = \frac{\lambda_0}{\gamma\sigma(Y_t)} X_t.$$

- uncorrelated case $\rho = 0$, the problem is "linear" since $q = 1$

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} \mathbb{E}\left[e^{\frac{1-\gamma}{2\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s} \middle| \mathcal{G}_t \right], \quad \pi_t^* = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} X_t.$$

Sketch of Proof (Verification)

- V_t is a supermartingale for any admissible control π
- V_t is a true martingale following π^*
- π^* is admissible

Define $\alpha_t = \pi_t / X_t$, then

$$\mathrm{d}V_t = V_t D_t(\alpha_t)\,\mathrm{d}t + \mathrm{d}\,\text{Martingale}$$

with the drift factor $D_t(\alpha_t)$

$$D_t(\alpha_t) := \alpha_t \mu - \frac{\gamma}{2}\alpha_t^2\sigma^2 - \frac{\lambda^2}{2\gamma} + \frac{q}{1-\gamma}a_t\xi_t + \frac{q(q-1)}{2(1-\gamma)}\xi_t^2 + \rho q \alpha_t \sigma \xi_t.$$

$\Rightarrow \alpha_t^*$ and $D_t(\alpha_t^*) = 0$ with the right choice of a_t and q:

$$a_t = -\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_t), \quad q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2}.$$

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$\Rightarrow$ α_t^* and $D_t(\alpha_t^*) = 0$ with the right choice of a_t and q:

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Multiple Assets Modeling

Consider the following model of S_t^1, S_t^2, ..., S_t^n

$$\mathrm{d}S_t^i = \mu^i(Y_t^i)S_t^i \,\mathrm{d}t + \sum_{j=1}^n \sigma_{ij}(Y_t^i)S_t^i \,\mathrm{d}W_t^j, \quad i = 1, 2, \dots n.$$

Each S_t^i is driven by a stochastic factor Y_t^i, but all factors Y_t^i are adapted to the same single Brownian motion W_t^Y with the correlation structure:

$$\mathrm{d}\left\langle W^i, W^j \right\rangle_t = 0, \quad \mathrm{d}\left\langle W^i, W^Y \right\rangle_t = \rho \,\mathrm{d}t, \quad \forall\, i, j = 1, 2, \dots, n, \quad n\rho^2 < 1.$$

Martingale Distortion Transformation with Multiple Assets

Then, the portfolio value V_t can be expressed as

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \mu(\mathbf{Y}_s)^\dagger \Sigma(\mathbf{Y}_s)^{-1}\mu(\mathbf{Y}_s)\,\mathrm{d}s} \,\middle|\, \mathcal{G}_t\right)\right]^q,$$

the constant q is chosen to be:

$$q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2 n}.$$

The optimal control π^* is given by

$$\pi_t^* = \left[\frac{\Sigma(\mathbf{Y}_t)^{-1}\mu(\mathbf{Y}_t)}{\gamma} + \frac{\rho q \xi_t \sigma^{-1}(\mathbf{Y}_t)^\dagger \mathbb{1}_n}{\gamma}\right] X_t.$$

Fractional Processes

A fractional Brownian motion $W_t^{(H)}$, $H \in (0,1)$

- a continuous Gaussian process
- zero mean
- $\mathbb{E}\left[W_t^{(H)} W_s^{(H)}\right] = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$
- $H < 1/2$: short-range correlation; $H > 1/2$: long-range correlation

A fractional Ornstein–Uhlenbeck process solves

$$\mathrm{d}Z_t^H = -aZ_t^H \,\mathrm{d}t + \mathrm{d}W_t^{(H)}$$

- stationary solution $Z_t^H = \int_{-\infty}^t e^{-a(t-s)} \,\mathrm{d}W_s^{(H)} = \int_{-\infty}^t \mathcal{K}(t-s) \,\mathrm{d}W_s^Z$
- Gaussian process with zero mean and constant variance
- $\mathcal{K}$ is non-negative, $\mathcal{K} \in L^2$

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Merton Problem under Slowly Varying Fractional SV

Consider a rescaled stationary fOU process $Z_t^{\delta,H}$

$$\begin{cases} \mathrm{d}S_t = S_t\left[\mu(Z_t^{\delta,H})\,\mathrm{d}t + \sigma(Z_t^{\delta,H})\,\mathrm{d}W_t\right], \\ Z_t^{\delta,H} = \int_{-\infty}^t \mathcal{K}^\delta(t-s)\,\mathrm{d}W_s^Z, \quad \mathcal{K}^\delta(t) = \sqrt{\delta}\mathcal{K}(\delta t), \end{cases} \quad \mathrm{d}\left\langle W, W^Z\right\rangle_t = \rho\,\mathrm{d}t.$$

Our study gives, for all $H \in (0,1)$:

- The value process $V_t^\delta := \operatorname{ess\,sup}_{\pi\in\mathcal{A}_t^\delta} \mathbb{E}\left[U(X_T^\pi)\middle|\,\mathcal{F}_t\right]$
- The corresponding optimal strategy π^*
- First order approximations to V_t^δ and π^*
- A practical strategy to generate this approximated value process

Apply the martingale distortion transformation with $Y_t = Z_t^{\delta,H}$

$$V_t^\delta = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Z_s^{\delta,H})\,\mathrm{d}s}\middle|\,\mathcal{G}_t\right)\right]^q,$$

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Approximation to the Value Process

Theorem (Fouque-H. '17)

For fixed $t \in [0, T)$, the value process V_t^δ takes the form

$$V_t^\delta = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} + \frac{X_t^{1-\gamma}}{\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} \lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\phi_t^\delta$$
$$+ \delta^H \rho \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} \lambda^2(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H}) \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})}$$
$$+ \mathcal{O}(\delta^{2H}),$$

where ϕ_t^δ is the random component of order δ^H

$$\phi_t^\delta = \mathbb{E}\left[\int_t^T \left(Z_s^{\delta,H} - Z_0^{\delta,H}\right) \mathrm{d}s \,\middle|\, \mathcal{G}_t\right].$$

Approximation to the Optimal Strategy

Recall that

$$\pi_t^* = \left[\frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} + \frac{\rho q \xi_t}{\gamma\sigma(Z_t^{\delta,H})}\right] X_t$$

and ξ_t is from the martingale rep. of $M_t = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_0^T \lambda^2(Z_s^{\delta,H})\,\mathrm{d}s} \Big| \mathcal{G}_t\right]$.

Theorem (Fouque-H., '17)

The optimal strategy π_t^* is approximated by

$$\pi_t^* = \left[\frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} + \delta^H \frac{\rho(1-\gamma)}{\gamma^2\sigma(Z_t^{\delta,H})}\frac{(T-t)^{H+1/2}}{\Gamma(H+\frac{3}{2})}\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\right] X_t + \mathcal{O}(\delta^{2H})$$

$$:= \pi_t^{(0)} + \delta^H \pi_t^{(1)} + \mathcal{O}(\delta^{2H}).$$

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How Good is the Approximation?

Corollary

In the case of power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\pi^{(0)} = \frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})}X_t$ generates the approximation of V_t^δ up to order δ^H (leading order + two correction terms of order δ^H), thus asymptotically optimal in $\mathcal{A}_t^\delta$.

- $H = \frac{1}{2}$, $Z_t^{\delta,H}$ becomes the Markovian OU process, both approximation coincides with results in [Fouque Sircar Zariphopoulou '13]. The corollary recovers [Fouque -H. '16].
- Sketch of proofs: Apply Taylor expansion to $\lambda(z)$ at the point $Z_0^{\delta,H}$, and then control the moments $\left(Z_t^{\delta,H} - Z_0^{\delta,H}\right) \sim \delta^H$.

How Good is the Approximation?

Corollary

In the case of power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\pi^{(0)} = \frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} X_t$ generates the approximation of V_t^δ up to order δ^H (leading order + two correction terms of order δ^H), thus asymptotically optimal in $\mathcal{A}_t^\delta$.

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Merton Problem under Fast-Varying Fractional SV

Consider a ϵ-scaled stationary fOU process $Y_t^{\epsilon,H}$

$$Y_t^{\epsilon,H} = \epsilon^{-H} \int_{-\infty}^{t} e^{-\frac{a(t-s)}{\epsilon}} \,\mathrm{d}W_s^{(H)} = \int_{-\infty}^{t} \mathcal{K}^{\epsilon}(t-s)\,\mathrm{d}W_s^{Y}, \mathcal{K}^{\epsilon}(t) = \frac{1}{\sqrt{\epsilon}}\mathcal{K}(\frac{t}{\epsilon})$$

together with the risky asset

$$\mathrm{d}S_t = S_t\left[\mu(Y_t^{\epsilon,H})\,\mathrm{d}t + \sigma(Y_t^{\epsilon,H})\,\mathrm{d}W_t\right], \quad \mathrm{d}\left\langle W, W^Y\right\rangle_t = \rho\,\mathrm{d}t.$$

For power utilities, we obtain:

- The value process V_t^{ϵ} and the corresponding optimal strategy π^*
- First order approximations to V_t^{ϵ} and π^*
- A strategy $\pi^{(0)}$ to generate this approximated value process

Using ergodic property of $Y_t^{\epsilon,H}$, but only valid for $H \in (\frac{1}{2}, 1)$

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Approximation to the Value Process V_t^ϵ

Theorem (Fouque-H. '17)

For fixed $t \in [0, T)$, the value process V_t^ϵ takes the form

$$V_t^\epsilon = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} + \frac{X_t^{1-\gamma}}{\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \phi_t^\epsilon$$
$$+ \epsilon^{1-H} \rho \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \widetilde{\lambda} \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\langle \lambda\lambda' \rangle (T-t)^{H+\frac{1}{2}}}{a\Gamma(H+\frac{3}{2})}$$
$$+ o(\epsilon^{1-H}),$$

where ϕ_t^ϵ is the random component of order ϵ^{1-H}

$$\phi_t^\epsilon = \mathbb{E}\left[\frac{1}{2}\int_t^T \left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right) \mathrm{d}s \,\middle|\, \mathcal{G}_t\right].$$

Optimal Portfolio

Theorem (Fouque-H., '17)

The optimal strategy π_t^* *is approximated by*

$$\pi_t^* = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \epsilon^{1-H} \frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})} \frac{\langle \lambda\lambda' \rangle (T-t)^{H-1/2}}{a\Gamma(H+\frac{1}{2})} \right] X_t + o(\epsilon^{1-H})$$

Corollary

In the case of power utility, $\pi^{(0)} = \frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} X_t$ generates the approximation of V^ϵ up to order ϵ^{1-H} (leading order + two correction terms of order ϵ^{1-H}), thus asymptotically optimal in $\mathcal{A}_t^\epsilon$.

Ergodicity of $Y_t^{\epsilon,H}$

For $H > 1/2$, under appropriate assumption of $\lambda(\cdot)$,

$$\int_0^t (\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2)\,\mathrm{d}s,$$
$$\int_0^t (\lambda(Y_s^{\epsilon,H}) - \widetilde{\lambda})\,\mathrm{d}s,$$
$$\int_0^t (\lambda(Y_s^{\epsilon,H})\lambda'(Y_s^{\epsilon,H}) - \langle \lambda\lambda' \rangle)\,\mathrm{d}s,$$

are small and of order ϵ^{1-H}.

Comparison with the Markovian Case

- The value function and the optimal strategy are derived in [FSZ, '13]:

$$V^\epsilon(t, X_t) = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \left[1 - \sqrt{\epsilon}\rho\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\langle \lambda\theta' \rangle}{2}(T-t)\right] + \mathcal{O}(\epsilon)$$

$$\pi^*(t, X_t, Y_t^{\epsilon,H}) = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \sqrt{\epsilon}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\theta'(Y_t^{\epsilon,H})}{2}\right] X_t + \mathcal{O}(\epsilon)$$

- Formally let $H \downarrow \frac{1}{2}$ in our results:

$$V_t^\epsilon = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \left[1 + \sqrt{\epsilon}\rho\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\widetilde{\lambda}\,\langle \lambda\lambda' \rangle}{a}(T-t)\right] + o(\sqrt{\epsilon})$$

$$\pi_t^* = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \sqrt{\epsilon}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\langle \lambda\lambda' \rangle}{a}\right] X_t + o(\sqrt{\epsilon})$$

Work in preparation

- Rough Fast-varying fSV

$$Y_t^{\epsilon,H} = \int_{-\infty}^{t} \mathcal{K}^\epsilon(t-s)\,\mathrm{d}W_s^Y, \quad H < \frac{1}{2}$$

Surprisingly, $Y_t^{\epsilon,H}$ is not visible to the leading order nor in the first order correction:

$$V_t^\epsilon = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \left[1 + \sqrt{\epsilon}\rho\left(\frac{1-\gamma}{\gamma}\right)^2 \overline{D}(T-t)\right] + o(\sqrt{\epsilon})$$

- Multiscale fSV

$$\mathrm{d}S_t = S_t\left[\mu(Y_t^{\epsilon,H}, Z_t^{\delta,H'})\,\mathrm{d}t + \sigma(Y_t^{\epsilon,H}, Z_t^{\delta,H'})\,\mathrm{d}W_t\right],$$

$$Y_t^{\epsilon,H} = \int_{-\infty}^{t} \mathcal{K}^\epsilon(t-s)\,\mathrm{d}W_t^Y, \quad Z_t^{\delta,H'} = \int_{-\infty}^{t} \mathcal{K}^\delta(t-s)\,\mathrm{d}W_t^Z.$$

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Merton Problem under General Utility

Martingale Distortion Transformation is not available
$\rightarrow$ Start with a given strategy $\pi^{(0)}$

- A first order approximation to $V^{\pi^{(0)},\delta}$
  obtained by epsilon-martingale decomposition[3][4]
- Optimality of $\pi^{(0)}$ in a smaller class of controls of feedback form

Denote by $v^{(0)}(t,x,z)$ the value function at the Sharpe-ratio $\lambda(z)$, we define $\pi^{(0)}$ by

$$\pi^{(0)}(t,x,z) = -\frac{\lambda(z)}{\sigma(z)} \frac{v_x^{(0)}(t,x,z)}{v_{xx}^{(0)}(t,x,z)}$$

and the associated value process $V^{\pi^{(0)},\delta}$

$$V_t^{\pi^{(0)},\delta} := \mathbb{E}\left[U(X_T^{\pi^{(0)}})|\mathcal{F}_t\right].$$

[3] Fouque Papanicolaou Sircar '01
[4] Garnier Solna '15

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Epsilon-Martingale Decomposition

Finding $Q_t^{\pi^{(0)},\delta}$ such that

- $Q_T^{\pi^{(0)},\delta} = V_T^{\pi^{(0)},\delta} = U(X_T^{\pi^{(0)}})$,
- $Q_t^{\pi^{(0)},\delta} = M_t^\delta + R_t^\delta$, where M_t^δ is a martingale and R_t^δ is of order δ^{2H}.

Then

$$\begin{aligned} V_t^{\pi^{(0)},\delta} &= \mathbb{E}\left[Q_T^{\pi^{(0)},\delta} | \mathcal{F}_t\right] = M_t^\delta + \mathbb{E}\left[R_T^\delta | \mathcal{F}_t\right] \\ &= Q_t^{\pi^{(0)},\delta} - R_t^\delta + \mathbb{E}\left[R_T^\delta | \mathcal{F}_t\right], \end{aligned}$$

and $Q_t^{\pi^{(0)},\delta}$ is the approximation to $V^{\pi^{(0)},\delta}$ with error of order $\mathcal{O}(\delta^{2H})$

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First order approximation to $V^{\pi^{(0)},\delta}$

Proposition

For fixed $t \in [0, T)$, the $\mathcal{F}_t$-measurable value process $V_t^{\pi^{(0)},\delta}$ is of the form

$$V_t^{\pi^{(0)},\delta} = Q_t^{\pi^{(0)},\delta}(X_t^{\pi^{(0)}}, Z_0^{\delta,H}) + \mathcal{O}(\delta^{2H}),$$

where $Q_t^{\pi^{(0)},\delta}(x,z)$ is given by:

$$Q_t^{\pi^{(0)},\delta}(x,z) = v^{(0)}(t,x,z) + \lambda(z)\lambda'(z)D_1 v^{(0)}(t,x,z)\phi_t^\delta + \delta^H \rho\lambda^2(z)\lambda'(z)D_1^2 v^{(0)}(t,x,z)\frac{(T-t)^{H+3/2}}{\Gamma(H+\frac{5}{2})}.$$

- For power utility, $Q_t^{\pi^{(0)},\delta}$ coincides with the approximation of V_t^δ
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Asymptotically Optimality of $\pi^{(0)}$

Theorem (Fouque-H. '17)

The trading strategy $\pi^{(0)}(t,x,z) = -\frac{\lambda(z)}{\sigma(z)}\frac{v_x^{(0)}(t,x,z)}{v_{xx}^{(0)}(t,x,z)}$ *is asymptotically optimal in the following class:*

$$\widetilde{\mathcal{A}}_t^\delta[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha] := \left\{\pi = \widetilde{\pi}^0 + \delta^\alpha \widetilde{\pi}^1 : \pi \in \mathcal{A}_t^\delta, \alpha > 0, 0 < \delta \leq 1\right\}.$$

Thank you !

Stochastic Volatility with Fast Factor Y_t

S_t is modeled by:

$$\begin{cases} \mathrm{d}S_t = \mu(Y_t)S_t\,\mathrm{d}t + \sigma(Y_t)S_t\,\mathrm{d}W_t, \\ \mathrm{d}Y_t = \frac{1}{\epsilon}b(Y_t)\,\mathrm{d}t + \frac{1}{\sqrt{\epsilon}}a(Y_t)\,\mathrm{d}W_t^Y, \end{cases}$$

with correlation $\mathrm{d}W_t W_t^Y = \rho\,\mathrm{d}t$.

Theorem (Fouque-H., in prep.)

Under appropriate assumptions, for fixed (t,x,y) and any family of trading strategies $\mathcal{A}_0(t,x,y)\left[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha\right]$, the following limit exists and satisfies

$$\ell := \lim_{\epsilon \to 0} \frac{\widetilde{V}^\epsilon(t,x,y) - V^{\pi^{(0)},\epsilon}(t,x,y)}{\sqrt{\epsilon}} \leq 0.$$

Theorem (Fouque-H., in prep.)

The residual function $E(t,x,y) := V^{\pi^{(0)},\epsilon}(t,x) - v^{(0)}(t,x) - \sqrt{\epsilon}v^{(1)}(t,x)$ *is of order* ϵ, *where in this case,* $v^{(0)}$ *solves*

$$v_t^{(0)} - \frac{1}{2}\overline{\lambda}^2\frac{\left(v_x^{(0)}\right)^2}{v_{xx}^{(0)}} = 0,$$

and $v^{(1)} = -\frac{1}{2}(T-t)\rho_1 B D_1^2 v^{(0)}(t,x)$.