Modelling anticipations in financial markets

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UCONN

Motivation

Financial markets obviously have asymmetry of information. That is, there are different type of traders whose behavior is induced by different types of information that they possess. Let us consider a "small" investor who trades in a arbitrage free financial market so as to maximize the expected utility of his wealth at a given time horizon. We assume that he possesses extra information about the future price of a stock.

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Financial markets obviously have asymmetry of information. That is, there are different type of traders whose behavior is induced by different types of information that they possess. Let us consider a "small" investor who trades in a arbitrage free financial market so as to maximize the expected utility of his wealth at a given time horizon. We assume that he possesses extra information about the future price of a stock.

Our basic question is: What is the value of this information ?

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Namely, we assume that there is a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}\right)$ which satisfies the usual conditions (i.e. $\mathcal{F}$ is complete and right-continuous and $\mathcal{F}_0$ is assumed to be trivial) and which is such that the price process of a given contingent claim is a continuous $d-$dimensional and $\mathcal{F}$-adapted square integrable martingale $(S_t)_{0\le t\le T}$.

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Since S is a martingale under $\mathbb{P}$, the market is a zero-sum game under the probability $\mathbb{P}$.

Informed insiders

Let $\mathcal{P}$ be a Polish space (for example $\mathcal{P} = \mathbb{R}^n$, $\mathcal{P} = C(\mathbb{R}_+, \mathbb{R}^n)$, etc...) endowed with its Borel $\sigma-$algebra $\mathcal{B}(\mathcal{P})$ and let $Y : \Omega \to \mathcal{P}$ be an $\mathcal{F}_T-$ measurable random variable: it will be a functional of the trajectories of the price process about which the insider has some information.

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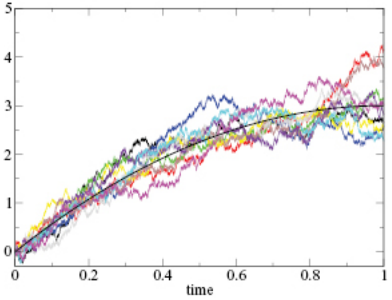

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We denote by $\mathbb{P}_Y$ the law of Y and assume that Y admits a regular disintegration with respect to the filtration $\mathcal{F}$

$$\mathbb{P}\left(Y \in dy \mid \mathcal{F}_t\right) = \eta_t^y\ \mathbb{P}_Y\left(dy\right).$$

Weakly informed insiders

Let $\mathcal{E}^{\nu}$ be the set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ such that:

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<u>Note:</u> $\mathcal{E}^{\nu_1} \cap \mathcal{E}^{\nu_2} = \mathcal{E}^{\nu_1 \otimes \nu_2}$.

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An insider performing portfolio optimization under a probability $\mathbb{Q}$ is called a weakly informed insider.

Utility functions

Definition

A utility function is a strictly increasing, strictly concave and twice continuously differentiable function

$$U : (0, +\infty) \to \mathbb{R}$$

which satisfies

$$\lim_{x \to +\infty} U'(x) = 0, \quad \lim_{x \to 0^+} U'(x) = +\infty.$$

Portfolio optimization problem

Problem

The insider's portfolio optimization problem is to find

$$\sup_{\Theta \in \mathcal{A}_{\mathcal{F}}(S)} \mathbb{E}^{\mathbb{Q}} \left(U \left(x + \int_0^t \Theta_u dS_u \right) \right)$$

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$x > 0$ *being the initial endowment of the insider.*

Value of the weak information

Definition

We define the financial value of the weak information (Y, ν) as being

$$u(x, \nu) := \inf_{\mathbb{Q} \in \mathcal{E}^{\nu}} \sup_{\Theta \in \mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}} \left(U \left(x + \int_0^T \Theta_u dS_u \right) \right).$$

In other terms, $u(x, \nu) - U(x)$ is the lowest increase in utility that can be gained by the insider from this extra knowledge .

Value of the weak information

Definition

The minimal probability is defined by

$$\mathbb{P}^{\nu} = \int_{\mathcal{P}} \mathbb{P}(\cdot | Y = y)\nu(dy).$$

Value of the weak information

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$$\begin{aligned} u(x,\nu) &= \sup_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\nu}\left(U\left(x+\int_0^T \Theta_u dS_u\right)\right) \\ &= \int_{\mathcal{P}} (U\circ I)\left(\frac{\Lambda(x)}{\xi(y)}\right)\nu(dy) \end{aligned}$$

where $\Lambda(x)$ is defined by

$$\int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\mathbb{P}_Y(dy) = x.$$

I is the inverse of U' and ξ is the density of ν w.r.t. $\mathbb{P}_Y$.

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Karatzas, I., Lehoczky, J.P., Shreve, S., 1987

Optimal wealth

Moreover, under $\mathbb{P}^{\nu}$ the optimal wealth process is given by

$$V_t = \int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right) \eta_t^y \mathbb{P}_Y(dy),$$

and the corresponding number of parts invested in the risky asset S by

$$\Theta_t = \int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right) \mathsf{D}_t \eta_t^y \mathbb{P}_Y(dy).$$

Examples of utility functions

1. Let $\alpha \in (0,1)$ and $U(x) = \frac{x^\alpha}{\alpha}$ then

$$u(x,\nu) = \frac{x^\alpha}{\alpha}\left[\int_{\mathcal{P}} \left(\frac{d\nu}{d\mathbb{P}_Y}(y)\right)^{\frac{1}{1-\alpha}} \mathbb{P}(Y \in dy)\right]^{1-\alpha}$$

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2. Let $U(x) = \ln x$ then

$$u(x,\nu) = \ln x + \int_{\mathcal{P}} \frac{d\nu}{d\mathbb{P}_Y}(y) \ln \frac{d\nu}{d\mathbb{P}_Y}(y)\,\mathbb{P}(Y \in dy).$$

Schrödinger processes

The minimal probability $\mathbb{P}^\nu$ associated to a weak anticipation is related to the theory of Schrödinger processes.

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Problem

Find a drift Θ so that the controlled process

$$Z_t^{\Theta} = \int_0^t \Theta_s ds + B_t$$

satisfies $Y(Z^{\Theta}) \sim \nu$ and $\mathbb{E}\left(\int_0^T \Theta_s^2 ds\right)$ is minimal.

Schrödinger processes

Solution

The law of Z^{Θ} is $\mathbb{P}^{\nu}$ and the corresponding information drift is given by the formula

$$\Theta_t = \mathsf{D}_t \ln \mathbb{E}\left(\xi(Y) \mid \mathcal{F}_t\right).$$

Example: Markovian bridges

If $Y = S_T$ and the dynamics of S under $\mathbb{P}$ follows a SDE

$$dS_t = \sigma(S_t)dB_t$$

Then

$$V_t = \int_{\mathbb{R}^n} I\left(\frac{\Lambda(x)}{\xi(y)}\right) p_{T-s}(S_t, y)dy,$$

and the corresponding number of parts invested in the risky asset S is given by

$$\Theta_t = \int_{\mathbb{R}^n} I\left(\frac{\Lambda(x)}{\xi(y)}\right) \nabla p_{T-s}(S_t, y)dy.$$

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In this Markovian case, the optimal proportion process solves the Burger's equation. More precisely

$$\Pi(t, S_t) = \nabla \ln \Phi(t, S_t)$$

where Φ solves

$$\frac{\partial \Phi}{\partial t} + L\Phi = 0$$

Anticipations on the Black-Scholes model

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In the market

$$\left(\Omega, (\mathcal{F}_t)_{t \leq T}, (S_t)_{t \leq T}, \mathbb{P}^{\nu}\right)$$

where ν is the distribution of B_T, the optimal wealth process is given by

$$V_t = h\left(t, B_t\right)$$

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where B satisfies

$$dB_t = \frac{\int_{-\infty}^{+\infty}\left(\frac{y-B_t}{T-t}\right) e^{\frac{y^2}{2T}-\frac{(y-B_t)^2}{2(T-t)}}\nu(dy)}{\int_{-\infty}^{+\infty} e^{\frac{y^2}{2T}-\frac{(y-B_t)^2}{2(T-t)}}\nu(dy)} dt + d\beta_t,\ \ t < T$$

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and h given by

$$h(t,y) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} I\left(\frac{\Lambda(x)}{\xi(z)}\right) e^{-\frac{(z-y)^2}{2(T-t)}} \, dz.$$

Gaussian example

Assume ν is a Gaussian:

$$\nu(dx) = \frac{e^{-\frac{(x-m)^2}{2s^2}}}{\sqrt{2\pi}s} dx$$

with $m \in \mathbb{R}$ and $s^2 \le T$.

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with $m \in \mathbb{R}$ and $s^2 \leq T$.
Let

$$\delta = \frac{s^2 - T}{T}$$

the relative variance ($\delta = 0$ corresponds to the Black and Scholes model).

Optimal strategies

The optimal proportion processes Π_t and the optimal expected utilities $u(x, \nu)$ for the utility functions U below are given as follows:

1. Logarithmic utility $U :]0, +\infty[\to \mathbb{R}$, $x \to \ln x$.

$$\Pi_t = \frac{1}{\sigma} \frac{\delta B_t + m}{\delta t + T} \text{ , } 0 \leq t \leq T,$$

$$u(x, \nu) = \ln x + \frac{1}{2} \left(\delta - \ln(1 + \delta) + \frac{m^2}{T} \right).$$

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2. Power utility $U :]0,+\infty[\to \mathbb{R}$, $x \to \frac{x^\alpha}{\alpha}$, $\alpha \in]0,1[$.

$$\Pi_t = \frac{1}{\sigma}\frac{\delta B_t + m}{\delta t + T(1-\alpha) - \alpha\delta T}\ , 0 \leq t \leq T,$$

$$u(x,\nu) = \frac{x^\alpha}{\alpha}\ \frac{1}{\sqrt{1+\delta}}\left(\frac{1-\alpha}{\frac{1}{1+\delta} - \alpha}\right)^{\frac{1-\alpha}{2}} \exp\left(\frac{\alpha m^2}{2(T(1-\alpha) - \alpha\delta T)}\right)$$

Incomplete markets

Theorem (B.-Nguyen-Ngoc 2004)

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Let us assume that the market is incomplete. Then for each initial investment $x > 0$,

$$u(x,\nu) = \inf_{y>0}\left(\inf_{\pi\in\mathcal{D}}\int_{\mathcal{P}} \tilde{U}(x\pi(u))\nu du + xy\right)$$

where

$$\mathcal{D} = \left\{\frac{d\mathbb{P}^*_Y}{d\nu}, \quad \mathbb{P}^* \in \mathcal{M}(S)\right\}$$

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We assume asymptotic elasticity of the utility

$$\lim_{x\to+\infty} x\frac{U'(x)}{U(x)} < 1$$

The notion of strong information

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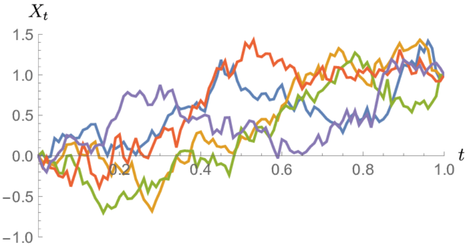

The notion of strong information

We study here the financial market

$$\left(\Omega, (\mathcal{G}_t)_{0 \le t < T}, \mathbb{P}, (S_t)_{0 \le t < T}\right)$$

where $\mathcal{G}_t = \sigma(\mathcal{F}_t, Y)$ is the initially enlarged filtration.

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where $\mathcal{G}_t = \sigma(\mathcal{F}_t, Y)$ is the initially enlarged filtration.
For $t < T$, there is no arbitrage in the time interval $[0, t]$. But there is an arbitrage in the time interval $[0, T]$.

Portfolio optimization problem

For $t \in [0, T]$, *the insider's portfolio optimization problem on* $[0, t]$ *is to find*

$$u(x, Y, t) := \sup_{\Theta \in \mathcal{A}_{\mathcal{G}}(S)} \mathbb{E}\left(U\left(x + \int_0^t \Theta_u dS_u \right) \right),$$

$x > 0$ *being the initial endowment of the insider.*

Let $t \in [0, T)$, $x > 0$ and let us assume that there exists an $\sigma(Y)-$measurable random variable $\Lambda_t(x) : \Omega \to (0, +\infty)$ with

$$\mathbb{E}\left(\frac{1}{\eta_t^Y} I \left(\frac{\Lambda_t(x)}{\eta_t^Y} \right) \mid Y \right) = x,$$

then

$$u(x, Y, t) = \mathbb{E}\left(\ (U \circ I) \left(\frac{\Lambda_t(x)}{\eta_t^Y} \right) \right).$$

Examples

Let $\alpha \in (0,1)$ and $U(x) = \frac{x^\alpha}{\alpha}$, then

$$I(y) = y^{\frac{1}{\alpha-1}},$$

$$\Lambda_t(x) = \frac{x^{\alpha-1}}{\mathbb{E}\left(\left(\eta_t^Y\right)^{\frac{\gamma}{1-\gamma}} \mid Y\right)},$$

and

$$u(x, Y, t) = \frac{x^\alpha}{\alpha}\mathbb{E}\left(\mathbb{E}\left[\left(\eta_t^Y\right)^{\frac{\alpha}{1-\alpha}} \mid Y\right]^{1-\alpha}\right).$$

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Let $U(x) = \ln x$, then

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$$u(x, Y, t) = \ln x + \mathbb{E}\left(\ln \eta_t^Y\right).$$

Further directions:

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- Dynamic flow of informations: *Dynamic Markov bridges motivated by models of insider trading*, L Campi, U Cetin, A Danilova, SPA 2011.