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Near-Expiry Asymptotics of the Implied Volatility in Local and Stochastic Volatility Models

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Black-Scholes PDE and Black-Scholes formula

Consider a simplified financial market with risk-free interest rate r and a single stock with volatility σ. A call option price function $C(s,t)$ satisfies the Black-Scholes PDE (1973) is

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + rsC_s - rC = 0, \qquad (t,s) \in (0,T) \times (0,\infty).$$

with the terminal condition $C(T,s) = (s-K)^+$.

The Black-Scholes equation has an explicit solution called the Black-Scholes formula.

$$C^{BS}(t,s;\sigma,r) = sN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2}\, dx,$$

and

$$d_1 = \frac{\log(s/K) + (r + \sigma^2/2)(T-t)}{\sqrt{T-t}\sigma}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Probabilstic Model

Let S_t be the stock price process and $B_t = e^{rt}$ the risk-free bond. Under the risk-neutral probability, S_t is described by the following stochastic differential equation

$$dS_t = S_t(r dt + \sigma dW_t).$$

Here r is the risk-free interest rate, σ a constant volatility and W_t a standard 1-dimensional Brownian motion. This equation has an explicit solution

$$S_T = S_0 \exp\left[\sigma B_T + \left(r - \frac{\sigma^2}{2}\right) T\right].$$

The price of an European call option at the expiry T is

$$C(S_T, T) = (S_T - K)^+.$$

It also has the following probabilistic representation

$$C(t, s; \sigma, r) = e^{-r(T-t)} \mathbb{E}\{(S_T - K)^+ | S_t = s\}.$$

Dupire's local volatility model

Constant volatility model is inadequate. Dupire (1994) introduced the local volatility model in which volatility depends on the stock price. The Black-Scholes PDE becomes The Black-Scholes equation is

$$C_t + \frac{1}{2}\sigma(s)^2 s^2 C_{ss} + rsC_s - rC = 0.$$

If we make a change of variable $x = \log s$, then the equation becomes

$$C_t + LC = 0,$$

where

$$L = \frac{1}{2}\eta(x)^2 \frac{d^2}{dx^2} + \left[r - \frac{1}{2}\eta(x)^2\right] \frac{d}{dx}$$

with $\eta(x) = \sigma(e^x)$. Let $p\,(t, x, y)$ be the heat kernel for $\partial_t - L$. Then

$$C(t, s) = \int_0^\infty (e^y - K)^+ p\,(T - t, \log s, \log y)\, dy.$$

Implied volatility in local volatility model

If $c(t, s)$ is the call price in the local volatility model when the stock price is s at time t, then the implied volatility $\hat{\sigma}$ is defined implicitly by

$$C(t, s) = C^{BS}(t, s; \hat{\sigma}(t, s), r).$$

The call price on the right side of the above equation is the one calculated by the classical Black-Scholes formula.

Assume that the local volatility is uniformly bounded from both above and below

$$0 < \underline{\sigma} \leq \sigma(s) \leq \overline{\sigma} < \infty.$$

The implied volatility $\hat{\sigma}(t, s)$ is well defined because the Black-Scholes pricing function $C^{BS}(t, s; \sigma, r)$ is an increasing function of the volatility σ:

$$\frac{\partial C}{\partial \sigma} = \frac{\sqrt{T - t}\, S}{\sqrt{2\pi}}\, e^{-d_1^2/2}.$$

Analysis of symptotic behaviors

The range of the parameters: $(t, s) \in (0, T) \times (0, \infty)$.

Asymptotics: investigate the boundary behavior of the implied volatility function $\hat{\sigma}(s, t)$:

(1) near expiry: $t \to T$;

(2) deeply in the money: $s \to \infty$;

(3) deeply out of the money: $s \to 0$.

We are concerned with (1). Recall that

$$C(t, s) = C^{BS}(t, s, \hat{\sigma}(t, s), r)$$

and $C(T, s) = (s - K)^+$. If $s < K$, both sides vanish as $t \uparrow T$.

How does $\hat{\sigma}(t, s)$ behave when $t \uparrow T$?

The nonlinear parabolic equation of Berestycki, Busca, and Florent

The implied volatility function $\hat{\sigma}(t, s)$ satisfies an quasi-linear partial differential equation:

$$2\tau\hat{\sigma}\hat{\sigma}_\tau = \sigma^2\tau\hat{\sigma}\hat{\sigma}_{xx} + \sigma^2\left(1 - x\frac{\hat{\sigma}_x}{\hat{\sigma}}\right)^2 - \frac{1}{4}\sigma^2\tau^2\hat{\sigma}^2\hat{\sigma}_x^2 - \hat{\sigma}^2$$

Here $x = \log\left(se^{-r\tau}/K\right)$ and $\tau = T - t$ and $\hat{\sigma}(x, \tau) = \hat{\sigma}(s, t)$, the implied volatility in the new variables.

Berestycki, Busca and Florent:

(1) Asymptotics and calibration of local volatility models, *Quantitative Finance*, Vol. 2, 61-69 (2002)).

(2) Computing the implied volatility in stochastic volatility models, *Comm. Pure and Appl. Math.*, Vol. LVII, 1352-1373 (2004).

Implied Volatility asymptotics - leading term

Solution of the quasi-linear BBF equation has nice comparison properties. For the equation itself in this special form and for the comparison principle to hold, the use of the terminal call price function $C(T,s) = (s-K)^+$ is crucial.

The function $\hat{\sigma}(s,t)$ is NOT singular near $t = T$. Using the comparison properties, BBF identified the leading term as $t \to T$ of the leading value of the implied volatility:

$$\lim_{t \uparrow T} \hat{\sigma}(t,s) = \left[\frac{1}{\log K - \log s} \int_s^K \frac{du}{u\sigma(u,T)} \right]^{-1}.$$

This serves as the initial condition for the BBF nonlinear parabolic equation.

BBF = Berestycki, Busca, and Florent

Implied volatility expansion

We expect an asymptotic expansion

$$\hat{\sigma}(s,t) \sim \hat{\sigma}(s,T) + \hat{\sigma}_1(s,T)(T-t) + \hat{\sigma}_2(s,T)(T-t)^2 + \cdots$$

for the solution of the volatility expansion. Besides $\hat{\sigma}(s,T)$, the first two terms $\hat{\sigma}_1(s,T)$ and $\hat{\sigma}_2(s,T)$ also have practical significance.

Calculating $\hat{\sigma}_1(s,T)$ is from the BBF nonlinear PDE is not feasible in practice. We adopt a different approach: going back to the linear parabolic equation.

The new method also shows that the special form of the call function $(s-K)^+$, important for the comparison results for the nonlinear parabolic PDE, is in fact of no significance for the leading term calculation. Only the support of the function is important.

Qualitative analysis

Recall that

$$C(t,s) = \int_0^\infty (e^y - K)^+ p\,(T-t, \log s, \log y)\, dy.$$

Since $t \uparrow T$, the problem should be related to the short time behavior of the heat kernel. We have Varadhan's formula

$$\lim_{t\to 0} t \log p\,(t,x,y) = -\frac{d(x,y)^2}{2},$$

where $d(x,y)$ is the Riemannian distance between x and y. In the local volatility case (one dimension) we have

$$d(x,y) = \int_x^y \frac{dz}{z\sigma(z)}.$$

Finer asymptotics of the implied volatility $\hat{\sigma}(t,s)$ depends on more delicate but well studied asymptotic expansion of the heat kernel.

Refined implied volatility behavior - first order deviation

Theorem For the implied volatility function we have

$$\hat{\sigma}(t,s) = \hat{\sigma}(T,s) + \hat{\sigma}_1(T,s)(T-t) + O((T-t)^2),$$

where (due to Berestycki, Busca, and Florent)

$$\hat{\sigma}(T,s) = \left[\frac{1}{\log K - \log s}\int_s^K \frac{du}{u\sigma(u)}\right]^{-1}$$

and

$$\hat{\sigma}_1(T,s) = \frac{\hat{\sigma}(T,s)^3}{(\log K - \log s)^2}\left[\log\frac{\sqrt{\sigma(s)\sigma(K)}}{\hat{\sigma}(T,s)} + r\int_s^K \left[\frac{1}{\sigma^2(u)} - \frac{1}{\hat{\sigma}^2(T,s)}\right]\frac{du}{u}\right].$$

Special case: $r = 0$

The first order correction is given by

$$\hat{\sigma}_1(T,s) = \frac{\hat{\sigma}(T,s)^3}{(\log K - \log s)^2} \log \frac{\sqrt{\sigma(s)\sigma(K)}}{\hat{\sigma}(T,s)}.$$

This case was obtained by Henry-Labordere using a heuristic argument (in the manner of theoretical physics).

Observations

(1) The first order deviation $\hat{\sigma}_1(T, s)$ of the implied volatility $\hat{\sigma}(t, s)$ depends only on the extremal values $\sigma(s)$ and $\sigma(K)$ of the local volatility function and a certain averaged deviation of the local volatility from its expected leading value $\hat{\sigma}(T, s)$. This means that the first order deviation is *numerically stable* with respect to the local volatility function $\sigma(s)$, which in practice cannot be calibrated very precisely.

(2) Further analysis shows that further higher order derivatives of $\hat{\sigma}(t, s)$ at expiry are not as stable, i.e., they will be sensitive to the derivatives of the local volatility.

Basic approach

We have

$$C(t,s) = C^{BS}(t,s,\hat{\sigma}(t,s),r)$$

and

$$C(t,s) = \int_0^\infty (e^y - K)^+ p\,(T-t, \log s, \log y)\, dy.$$

(1) Find the expansion of the heat kernel $p(\tau, x, y)$ as $\tau \downarrow 0$.

(2) Use the heat kernel expansion to calculate the expansion of $C(t,s)$.

(3) Use the explicit Black-Scholes formula to expand C^{BS}.

(4) Equate the two expansions and extra the information on $\hat{(}t,s)$.

Heat kernel asymptotics

Denote the transition density of X by $p(\tau, x, y)$. Then we have the following expansion as $\tau \downarrow 0$

$$p_X(\tau, x, y) \sim \frac{u_0(x,y)}{\sqrt{2\pi\tau}} e^{-\frac{d^2(x,y)}{2\tau}} \Big[1 + \sum_{n=1}^{\infty} H_n(x,y) \tau^n \Big].$$

$$d(x,y) = \int_x^y \frac{du}{\eta(u)}.$$

H_n are smooth functions of x and y and

$$u_0(x,y) = \eta^{\frac{1}{2}}(x) \eta^{-\frac{3}{2}}(y) \exp \Big[-\frac{1}{2}(y-x) + \int_x^y \frac{r}{\eta^2(v)} dv \Big].$$

(e.g., see S. A. Molchanov, Diffusion processes and Riemannian geometry, *Russian Math. Surveys*, **30**, no. 1 (1975), 1-63.)

The leading term of the call price in a local volatility model

Introduce the new variables

$$\tau = T - t \qquad x = \log s.$$

Consider the case $x < \log K$. For the rescaled call price function

$$v(\tau, x) = C(t, s)$$

we have as $\tau \downarrow 0$,

$$v(\tau, x) = \left[K u_0(x, \log K) \left(\frac{\eta(\log K)}{d(x, \log K)} \right)^2 + O(\tau) \right] \frac{\tau^2}{\sqrt{2\pi\tau}} e^{-\frac{d(x, \log K)^2}{2\tau}}.$$

Asymptotic behavior of the classical Black-Scholes pricing function

Let

$$V(\tau, x; \sigma, r) = C^{BS}(t, e^x; \sigma, r)$$

be the classical Black-Scholes call price function. Then we have as $\tau \downarrow 0$,

$$V(\tau, x; \sigma, r) \sim \frac{1}{\sqrt{2\pi}} \frac{K\sigma^3 \tau^{3/2}}{(\ln K - x)^2} \exp\left[-\frac{\ln K - x}{2} + \frac{r(\ln K - x)}{\sigma^2}\right] \exp\left[-\frac{(\ln K - x)^2}{2\tau\sigma^2}\right] + R(\tau, x; \sigma, r).$$

The remainder satisfies

$$|R(\tau, x; \sigma, r)| \leq C\, \tau^{5/2} \exp\left[-\frac{(\ln K - x)^2}{2\tau\sigma^2}\right],$$

where $C = C(x, \sigma, r, K)$ is uniformly bounded if all the indicated parameters vary in a bounded region.

Last step

Equate the two expansions $v(\tau, x) = V(\tau, x; \hat{\sigma}(t, s), r)$ and extract the first two terms in the asymptotic expansion for $\hat{\sigma}(t, s)$.

Summary

The implies volatility function is the solution of a nonlinear parabolic equation

$$2\tau\hat{\sigma}\hat{\sigma}_\tau = \sigma^2\tau\hat{\sigma}\hat{\sigma}_{xx} + \sigma^2\left(1 - x\frac{\hat{\sigma}_x}{\hat{\sigma}}\right)^2 - \frac{1}{4}\sigma^2\tau^2\hat{\sigma}^2\hat{\sigma}_x^2 - \hat{\sigma}^2$$

with the boundary condition

$$\hat{\sigma}(T, s) = \left[\frac{1}{\log K - \log s}\int_s^K \frac{du}{u\sigma(u)}\right]^{-1}.$$

By exploring the connection with the heat kernel expansion, we are able to find the first order deviation of the implied volatility from its leading value.

Stochastic volatility models

Stochastic volatility model:

$$S_t^{-1} dS_t = r\,dt + \sigma(S_t, y_t)\,dW_t^1,$$
$$dy_t = \theta(y_t)\,dt + \nu(y_t)\,d\tilde{W}_t.$$

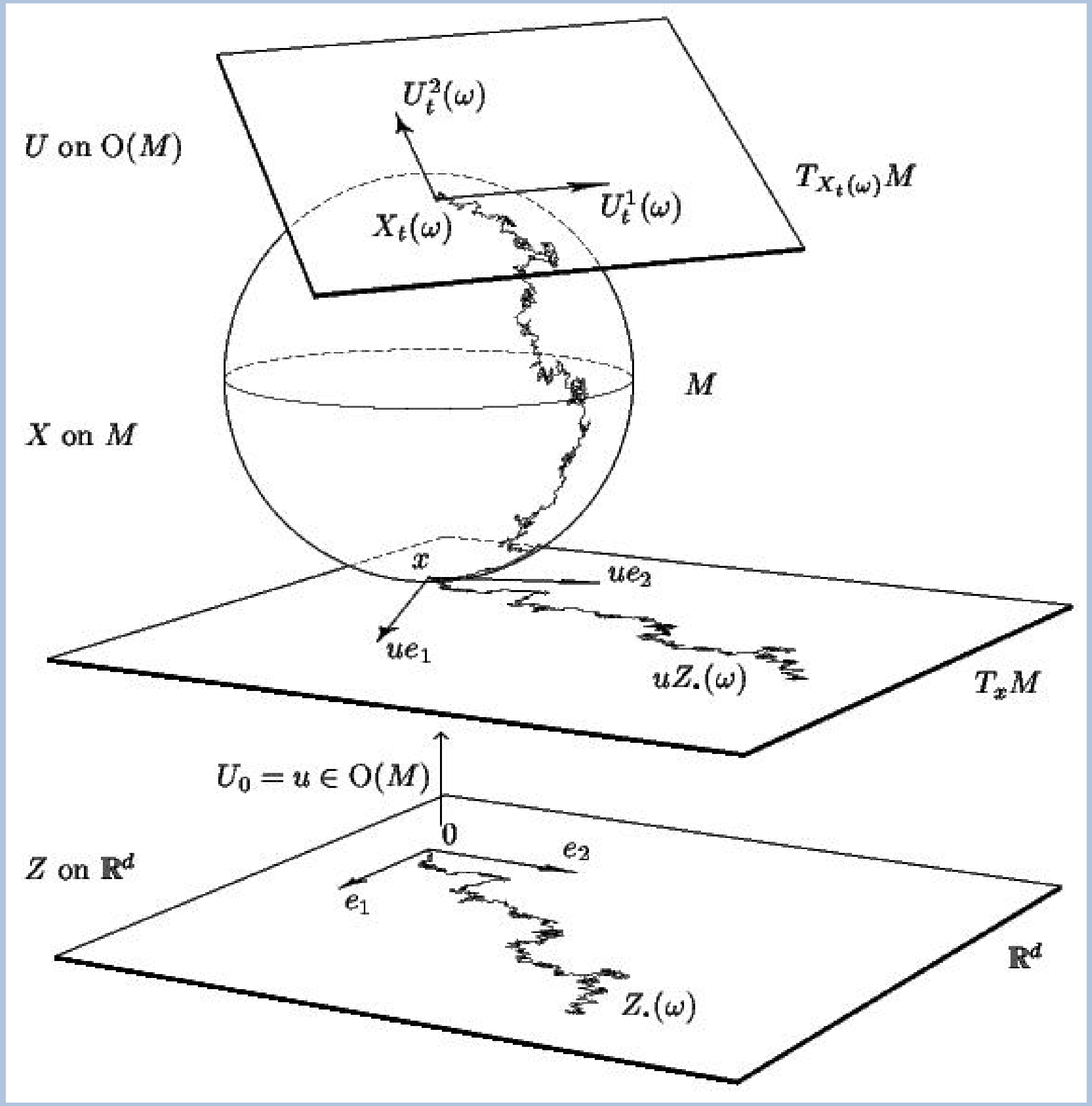
Here $W_t = (W_t^1, \tilde{W}_t) = (W_t^1, W_t^2, \cdots, W_t^n)$ is a linear transform of the standard Brownian motion. Let $Z_t = (S_t, y_t)$ be the combined stock-volatility process. The generator L of Z is a second order subelliptic operator. L determines a Riemannian geometry. A large number of stochastic volatility models lead to hyperbolic geometry (e.g., SABR models).

A similar result holds for the implied volatility:

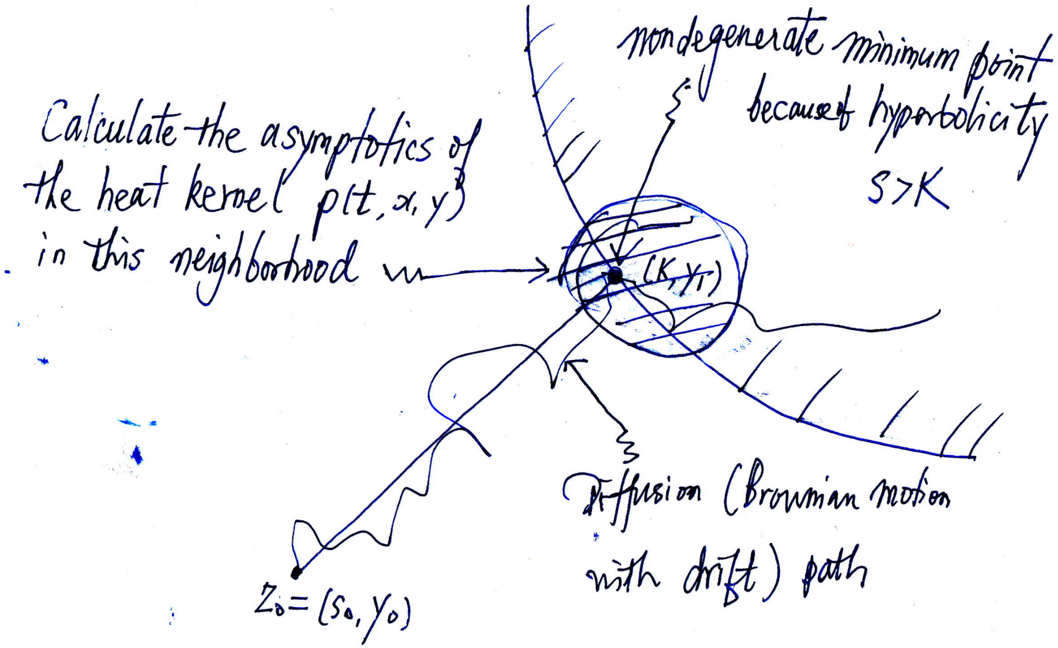
$$\hat{\sigma}(T, Z_0) = \frac{\ln S_0 - \ln K}{d(Z_0, H_K)},$$

where $d(Z_0, H_K)$ is the Riemannian distance of $Z_0 = (S_0, y_0)$ to the hypersurface $H_K = \{s = K\}$. First order deviation $\hat{\sigma}_1(T, Z_0)$ requires a detailed geometric analysis of the hypersurface H_K.

Brownian Motion on a Manifold: Rolling with slipping ... Source: Anton Thalmaier



Stochastic Volatility Model Localized Calculation Illustration



Localized Calculation for a Stochastic Volatility Model.

Thank You!