Approximate Hedging and BSDEs with weak boundaries

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based on joint works with
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Motivation

- Stock price : (with large investor's strategy π)

$$\frac{dS_u^\pi}{S_u^\pi} = \mu\left(u, S_u^\pi, \pi_u\right) du + \sigma\left(u, S_u^\pi, \pi_u\right) dW_u$$

- Wealth process : (risk free interest rate $r = 0$)

$$dX_u^\pi = \pi_u \frac{dS_u^\pi}{S_u^\pi} = \pi_u \left[\mu\left(u, S_u^\pi, \pi_u\right) du + \sigma\left(u, S_u^\pi, \pi_u\right) dW_u\right]$$

- Super Hedging problem of claim $h(S_T^\pi)$:

$$\inf\left\{x \geq 0 : \exists \pi \in \mathcal{A} \quad \text{s.t.} \quad X_T^{0,x,\pi} \geq h\left(S_T^\pi\right) \ \mathbb{P} - \text{ps}\right\} .$$

$\Longrightarrow$ Prudential approach which leads to expensive prices

- Quantile Hedging of the claim $h(S_T^\pi)$: Given $p \in (0,1)$, find

$$\inf\left\{x \geq 0 : \ \exists \pi \in \mathcal{A} \quad \text{s.t.} \quad \mathbb{P}\left[X_T^{t,x,\pi} \geq h\left(S_T^\pi\right)\right] \geq p\right\} .$$

How decreases the price when one accepts to keep some hedging risk ?

Agenda

Explicit solution in a complete market

- Restriction to a complete market (super-replication$\Leftrightarrow$ replication)

- Stock price under the (unique) Risk Neutral Measure $\mathbb{Q}$:

$$\frac{dS_u}{S_u} = \sigma(u, S_u)\, dW_u \qquad \text{(independent on } \pi\text{)}$$

- Wealth process :

$$dX_u^{\pi} = \pi_u \sigma(u, S_u)\ dW_u$$

- Dual problem reformulation :

Maximize the probability of hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_T^{0,x,\pi} \geq h(S_T)\right]$$

Föllmer and Leukert approach to quantile hedging

Maximize the probability of hedge for a given initial wealth x

$$\Updownarrow$$

$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_T^{0,x,\pi} \geq h(S_T)\right]$$

$$\Updownarrow$$

$$\max_{X \in L_T^0} \mathbb{P}\left[X \geq h(S_T)\right] \ \text{ under the constraint } \ \mathbb{E}^{\mathbb{Q}}\left[X\right] \leq x$$

$A = \{X \geq h(S_T)\}$ $\quad\Updownarrow\quad$ $X = h(S_T)\mathbf{1}_A$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}\left[A\right] \ \text{ under the constraint } \ \mathbb{E}^{\mathbb{Q}}\left[h(S_T)\mathbf{1}_A\right] \leq x$$

$\Updownarrow$ $\quad d\mathbb{Q}^h := \frac{h(S_T)}{\mathbb{E}^{\mathbb{Q}}[h(S_T)]} d\mathbb{Q}$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}\left[A\right] \ \text{ under the constraint } \mathbb{Q}^h[A] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}\left[h(S_T)\right]},$$

A interprets as the critical region while testing $\mathbb{Q}^h$ against $\mathbb{P}$.

Neyman-Pearson lemma $\Longrightarrow$ optimal critical region $A^*(x)$

Optimal strategy $\pi^*(x)$: the one which replicates $h(S_T)\mathbf{1}_{A^*(x)}$

Quantile replication price : $x^*(p)$ such that $\mathbb{P}[A^*(x^*(p))] = p$

Solution in General Case

- Pros :
  - Explicit solution in some simple (but important) cases.
  - Generic solution of the form : $X_T^{0,x,\pi} = h(S_T)\,\mathbf{1}_A$
  - Similar structure in incomplete markets.
- Cons :
  - Resolution of the dual problem
  - Explicit solution not known in general (numerics)
  - In incomplete markets, the dual problem is a control problem : how to solve it ?
  - Relies heavily on the duality between super-hedgeable claims and risk neutral measures.

$\Longrightarrow$ Alternative dynamic approach

The particular case of super-hedging

- The super hedging price at time 0

$$\inf \left\{ x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X_T^{0,x,\pi} \geq h\left(S_T^{\pi}\right)\right] = 1 \right\}$$

- Dynamic version of the super-hedging problem

$$v(t,s,1) \quad = \quad \inf \left\{ x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X_T^{t,x,\pi} \geq h\left(S_T^{t,s,\pi}\right)\right] = 1 \right\}$$

- Dual approach : $v(t,s;1) \; = \; \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[h\left(S_T^{t,s}\right)\right]$

- Direct approach of Soner and Touzi :
  - (DP1) : $x > v(t,s,1) \Rightarrow \exists\, \pi \in \mathcal{A}$ s.t. for all stopping time $\tau \leq T$

$$X_\tau^{t,x,\pi} \geq v(\tau, S_\tau^{t,s,\pi}, 1)$$

  - (DP2) : $x < v(t,s,1) \Rightarrow$ for all stopping time $\tau \leq T$ and $\pi \in \mathcal{A}$

$$\mathbb{P}\left[X_\tau^{t,x,\pi} > v(\tau, S_\tau^{t,s,\pi}, 1)\right] < 1$$

$\Rightarrow$ Allows to derive PDEs associated to $v(\cdot, 1)$.

A stochastic target approach to quantile hedging

- The quantile hedging price at time 0

$$\inf\left\{x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X_T^{0,x,\pi} \geq h\left(S_T^{\pi}\right)\right] \geq p\right\}$$

- Dynamic version of the super-hedging problem

$$v(t,s,p) \quad = \quad \inf\left\{x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X_T^{t,x,\pi} \geq h\left(S_T^{t,s,\pi}\right)\right] \geq p\right\}$$

- Non consistent dynamic problem
- Idea : consider the "probability of super-hedging" as a process $(P_s)_{s \leq t \leq T}$
- This process must be a martingale and therefore of the form

$$P_s^{t,p,\alpha} \quad = \quad p + \int_t^s \alpha_u dW_u \;, \qquad t \leq s \leq T \;, \qquad \text{with } \alpha \in \mathbf{L}^2$$

- The quantile hedging price rewrites

$$v(t,s,p) \quad = \quad \inf\left\{x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ and } \alpha \in \mathbf{L}^2 \text{ s.t. } \mathbf{1}_{X_T^{t,x,\pi} \geq h\left(S_T^{t,s,\pi}\right)} \geq P_T^{t,p,\alpha}\right\}$$

Dynamic programming for quantile replication

- Dynamic version of the quantile hedging price :

$$v(t,s,p) := \inf\left\{x \in \mathbb{R}\ ,\quad \exists\,(\pi,\alpha) \in \mathcal{A} \times \mathbf{L}^2 \quad \text{s.t.} \quad \mathbf{1}_{X_T^{t,x,\pi} \geq h(S_T^{t,s,\pi})} \geq P_T^{t,p,\alpha}\right\}$$

- Dynamic programming principle :

(DP1) : Starting with a wealth at time t greater than $v(t,s,p)$,
one can at any time $\tau \geq t$ be able to (P_τ)-quantile replicate :

$$x > v(t,s,p) \ \Rightarrow\ \exists(\pi^*,\alpha^*) \quad \text{s.t.} \quad X_\tau^{t,x,\pi^*} \geq v(\tau, S_\tau^{t,s,\pi^*}, P_\tau^{t,p,\alpha^*})\ , \quad \forall \tau \in [t,T]$$

(DP2) : Starting with a wealth at time t lower than $v(t,s,p)$,
it is impossible to quantile replicate :

$$x < v(t,s,p) \ \Rightarrow\ \forall(\pi,\alpha) \quad \mathbb{P}\left[X_\tau^{t,x,\pi} > v(\tau, S_\tau^{t,s,\pi}, P_\tau^{t,p,\alpha})\right] < 1\ , \quad \forall \tau \in [t,T]$$

Formal derivation of the Hamilton Jacobi Bellman equation

- Portfolio dynamics : $dX_r^\pi = \mu(r, S_r^\pi, \pi_r)\pi_r dr + \sigma(r, S_r^\pi, \pi_r)\pi_r dW_r$
- Dynamics of $v(., S_.^\pi, P_.^\alpha)$:

$$dv(r, S_r^\pi, P_r^\alpha) = \left[v_t + \mu S_r^\pi v_x + \frac{\sigma^2 S_r^\pi}{2} v_{xx} + \frac{\alpha^2}{2} v_{pp} + 2\alpha\sigma S_r^\pi v_{xp} \right] (r, S_r^\pi, P_r^\alpha) dr$$
$$+ \left[\sigma S_r^\pi v_x + \alpha_r v_p\right](r, S_r^\pi, P_r^\alpha) dW_r$$

- Take $x \sim v(t, s, p)$:

$$(DP1) \quad \Rightarrow \quad \exists(\pi^*, \alpha^*) \quad \text{s.t.} \quad X_\tau^{t,x,\pi^*} \geq v(\tau, S_\tau^{t,s,\pi^*}, P_\tau^{t,p,\alpha^*}), \quad \forall \tau \in [t, T]$$
$$(DP2) \quad \Rightarrow \quad \forall(\pi, \alpha) \quad \mathbb{P}\left[X_\tau^{t,x,\pi} > v(\tau, S_\tau^{t,s,\pi}, P_\tau^{t,p,\alpha})\right] < 1, \quad \forall \tau \in [t, T]$$

- Formally, we deduce the following HJB equation

$$\sup_{(\alpha,\pi)} \mu\pi - \left[v_t + \mu s v_s + \frac{\sigma^2 s}{2} v_{ss} + \frac{\alpha^2}{2} v_{pp} + 2\alpha\sigma s v_{sp} \right] (t, s, p) = 0$$

under the constraint $\quad \sigma\pi = [\sigma s v_s + \alpha v_p](t, s, p)$

Rigorous derivation

- PDE dynamics in the domain :

$$\sup_{(\alpha,\pi)} \mu\pi - \left[v_t + \mu s v_s + \frac{\sigma^2 s}{2} v_{ss} + \frac{\alpha^2}{2} v_{pp} + 2\alpha\sigma s v_{sp} \right](t,s,p) = 0$$

under the constraint $\quad \sigma\pi = [\sigma s v_s + \alpha v_p](t,s,p)$

- Main technical difficulty : the auxiliary control α is not bounded.
- The auxiliary control α is directly related to the primal control π.
- Boundary conditions :

$$\begin{aligned} \text{at } p = 0+ : &\quad v(t,s,0) \;=\; 0 \\ \text{at } p = 1- : &\quad v(t,s,1) \text{ is the super-replication price} \\ \text{at } t = T- : &\quad v(T,s,p) \;=\; pg(s) \end{aligned}$$

- Possible numerical approximation of the solution via PDE scheme

Explicit resolution in the Black Scholes model

- PDE in the Black Scholes model :

$$v_t + \frac{\sigma^2 s^2}{2} v_{ss} - \frac{\sigma^2 s^2}{2} \frac{|v_{sp}|^2}{v_{pp}} - \frac{\mu^2}{2\sigma^2} \frac{v_p^2}{v_{pp}} + \mu s \frac{v_p v_{sp}}{v_{pp}} = 0 \quad \text{with } v(T,s,p) = pg(s)$$

- Introduction of the Fenchel-Legendre transform $\tilde{v}(t,s,.)$ of $v(t,s,.)$:

$$\tilde{v}(t,s,y) \quad := \quad \sup_{p \in [0,1]} \; pq - v(t,s,p)$$

- The Fenchel Legendre transform $\tilde{v}$ "solves" the following linear PDE

$$\tilde{v}_t + \frac{\sigma^2 s^2}{2} \tilde{v}_{ss} + \mu s q \tilde{v}_{sq} + \frac{\mu^2}{2\sigma^2} q^2 \tilde{v}_{qq} = 0 \qquad \text{with } \tilde{v}(T,s,q) = (q - g(s))^+$$

- We deduce the probabilistic representation :

$$\tilde{v}(t,s,q) = \mathbb{E}[(Q_T^{t,q} - h(S_T^{t,s}))^+] \quad \text{with} \quad Q_.^{t,q} := q + \int_t^{.} \frac{\mu}{\sigma} Q_s^{t,q} dW_s$$

- We retrieve v by re-applying the Fenchel transform.

Extensions

- On the Dynamics :

$$S^\pi = s + \int_t^\cdot \mu\left(S_u^\pi, \pi_u\right) du + \int_t^\cdot \sigma\left(S_u^\pi, \pi_u\right) dW_u$$
$$X^\pi = x + \int_t^\cdot \rho\left(S_u^\pi, X_u^\pi, \pi_u\right) du + \int_t^\cdot \beta\left(S_u^\pi, X_u^\pi, \pi_u\right) dW_u$$

- On the Problems : Given ℓ: $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ and $p \in \operatorname{Im}(\ell)$,

$$v(t,s;p) := \inf\left\{x \in \mathbb{R} \;:\; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S_T^{t,s,\pi}, X_T^{t,x,\pi}\right)\right] \geq p\right\}.$$

- Possible range of applications

$$\ell(s,x) = \mathbf{1}\{x \geq g(s)\} \quad \Rightarrow \quad \text{Quantile Hedging}$$
$$\ell(s,x) = U([x-g(s)]^+) \text{ with } U \nearrow \text{ concave} \quad \Rightarrow \quad \text{Loss function}$$
$$\ell(s,x) = U(x-g(s)) \text{ with } U \nearrow \text{ concave} \quad \Rightarrow \quad \text{Indifference pricing}$$

- Dynamic programming based on the reformulation

$$v(t,s;p) = \inf\left\{x \in \mathbb{R}_+ :\; \exists\, (\pi,\alpha) \in \mathcal{A} \times \mathbf{L}^2 \text{ s.t. } \ell\left(S_T^{t,s,\pi}, X_T^{t,x,\pi}\right) \geq P_T^{t,p,\alpha}\right\}.$$

(good) leads for extensions...

- Utility maximization under quantile hedging type constraint :
  $\Longrightarrow$ PDE characterization but no numerics (at that point)

- Combination of several constraints :
  Given $\ell_1, \ell_2, \ldots, \ell_m$ and $p_i \in \mathrm{Im}(\ell_i)$ for $i \leq m$,
  $$v(t,s;p) \quad := \quad \inf\left\{x \in \mathbb{R} \ : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell_i\left(S_T^{t,s,\pi}, X_T^{t,x,\pi}\right)\right] \geq p_i \ , \ \forall i \leq m\right\}$$
  $\Longrightarrow$ leads to high dimensional PDE, impossible to solve numerically

- Robust quantile hedging under model uncertainty
  Given a class of model $(\mathbb{P}^\lambda)_\lambda$, try to quantile hedge in any model
  $$\inf\left\{x \in \mathbb{R} \ : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}^{\mathbb{P}^\lambda}\left[\ell\left(\lambda, S_T^{t,s,\pi}, X_T^{t,x,\pi}\right)\right] \geq p_\lambda \ , \qquad \forall \lambda\right\} .$$
  $\Longrightarrow$ consider dynamic games

- One day ahead constraint :
  Given a time delay $\delta > 0$, try to find
  $$\inf\left\{x \in \mathbb{R} \ : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}_s\left[\ell\left(X_{s+\delta}^{t,x,\pi}\right)\right] \geq p \ , \qquad \forall s \leq T\right\} .$$
  $\Longrightarrow$ hard to get a dynamic programming principle

A BSDE approach to quantile hedging

- Consideration of non markovian terminal claim ξ.
- In a complete market, the replication price identifies as the solution of the BSDE (with no driver)

$$Y_t = Y_T - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T \qquad \text{with } Y_T = \xi$$

$\Longrightarrow$ Y price process and Z investment strategy (up to the volatility)

- In case of imperfections (e.g. portfolio constraints), the super-replication price of ξ identifies to the minimal solution to the BSDE

$$Y_t = Y_T - \int_t^T Z_s dW_s + \int_t^T dL_s\,, \quad 0 \leq t \leq T \qquad \text{with } Y_T \geq \xi$$

where L is an increasing process.

- For the quantile replication price, we expect

$$Y_T \geq \xi \qquad \text{to be replaced by} \qquad \mathbb{P}(Y_T \geq \xi) \geq p$$

BSDE with weak terminal condition

- Hence, this formally leads to a (no driver) new type of BSDE of the form

$$Y_t = Y_T - \int_t^T Z_s dW_s + \int_t^T dL_s \,, \quad 0 \le t \le T \qquad \text{with } \mathbb{P}(Y_T \ge \xi) \ge p$$

- More generally, for an increasing loss function ℓ, we get

$$dY_t = Z_t dW_t - dL_t \,, \qquad \text{with } \mathbb{E}[\ell(Y_T - \xi)] \ge p$$

- For a random increasing function ψ, we look towards the minimal solution to the new type of BSDE

$$dY_t = -g(t, Y_t, Z_t)dt + Z_t dW_t - dL_t \,, \qquad \text{with } \mathbb{E}[\psi(Y_T)] \ge p$$

- Constraint on the terminal condition distribution

$\Longrightarrow$ *" BSDE with weak terminal condition"*

BSDE with weak terminal condition

- For a random increasing function ψ and Lipschitz driver g, we look towards the minimal solution to the BSDE

$$dY_t \quad = \quad -g(t, Y_t, Z_t)dt + Z_t dW_t - dL_t \ , \qquad \text{with} \quad \mathbb{E}[\psi(Y_T)] \geq p$$

- Introduction of a supplementary control $\alpha \in \mathbf{L}^2$ and $P^{p,\alpha}_{\cdot} := p + \int_0^{\cdot} \alpha_s dW_s$

- Set of all possible terminal conditions : $(\psi^{-1}(P^{p,\alpha}_T))_{\alpha \in \mathbf{L}^2}$

- We suppose for simplicity $\psi : [0,1] \to [0,1]$

- Let $(Y^\alpha, Z^\alpha)_{\alpha \in \mathbf{L}^2}$ be the set of solutions to the classical BSDEs

$$dY^\alpha_t \quad = \quad -g(t, Y^\alpha_t, Z^\alpha_t)dt + Z^\alpha_t dW_t \ , \qquad \text{with} \quad Y^\alpha_T \ = \ \psi^{-1}(P^{p,\alpha}_T)$$

- At any time t, we can rewrite $Y^\alpha_t = \mathcal{E}^g_{t,T}\left[\psi^{-1}(P^{p,\alpha}_T)\right]$

Representation of the solution

- For $\alpha \in \mathbf{L}^2$, $(Y^\alpha = \mathcal{E}^g_{\cdot,T}\left[\psi^{-1}(P^{p,\alpha}_T)\right], Z^\alpha)$ solves the classical BSDE

$$dY^\alpha_t = -g(t, Y^\alpha_t, Z^\alpha_t)dt + Z^\alpha_t dW_t \,, \qquad \text{with} \quad Y^\alpha_T = \psi^{-1}(P^{p,\alpha}_T)$$

- Any Y-component of a super-solution to the BSDE with weak terminal condition is of the form Y^α.

- For any path α, in order to pay the cheapest price, we define :

$$\bar{Y}^\alpha_t := \operatorname{essinf}\left\{\mathcal{E}^g_{t,T}\left[\psi^{-1}(P^{p,\alpha'}_T)\right],\ \alpha' \in \mathbf{L}^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0,t]\right\}, \quad \forall t$$

- Obtention of Dynamic Programming Principle for the family $(\bar{Y}^\alpha)_\alpha$

$$\bar{Y}^\alpha_t = \operatorname{essinf}\left\{\mathcal{E}^g_{t,t'}\left[\bar{Y}^{\alpha'}_{t'}\right],\ \alpha' \in \mathbf{L}^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0,t]\right\}, \ 0 \le t \le t' \le T \,.$$

- $\bar{Y}^\alpha$ is indistinguishable from a ladlag g-submartingale

Representation of the solution

- For $\alpha \in \mathbf{L}^2$, $(Y^\alpha = \mathcal{E}^g_{.,T}\left[\psi^{-1}(P_T^{p,\alpha})\right], Z^\alpha)$ solves the classical BSDE

$$dY_t^\alpha = -g(t, Y_t^\alpha, Z_t^\alpha)dt + Z_t^\alpha dW_t \,, \qquad \text{with} \quad Y_T^\alpha = \psi^{-1}(P_T^{p,\alpha})$$

- For any path α, in order to pay the cheapest price, we define :

$$\bar{Y}_t^\alpha := \operatorname{essinf}\left\{\mathcal{E}^g_{t,T}\left[\psi^{-1}(P_T^{p,\alpha'})\right],\ \alpha' \in \mathbf{L}^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0,t]\right\}, \quad \forall t$$

- Additional assumption : $\psi^{-1}(\omega, .)$ is continuous for $\mathbb{P}$-a.e ω

$\Longrightarrow$ $\bar{Y}^\alpha$ is indistinguishable from a cadlag g-submartingale

- Characterization of the family $(\bar{Y}^\alpha)_{\alpha \in \mathbf{L}^2}$ of solutions :

Dynamics : $\bar{Y}_.^\alpha = \psi^{-1}(P_T^{p,\alpha}) + \int_.^T g(s, \bar{Y}_s^\alpha, \bar{Z}_s^\alpha)ds - \int_.^T \bar{Z}_s^\alpha dW_s + \bar{L}_.^\alpha - \bar{L}_T^\alpha$ on $[0,T]$

Minimality : $\bar{L}_{\tau_1}^\alpha = \operatorname{essinf}\left\{E\left[\bar{L}_{\tau_2}^{\alpha'}|\mathcal{F}_{\tau_1}\right],\ \alpha' \in \mathbf{L}^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0,\tau_1]\right\}, \quad \forall\, \tau_1 \le \tau_2$

Futur indep. : $\alpha' = \alpha$ on $[0,\tau]$ $\Longrightarrow$ $(\bar{Y}^{\alpha'}, \bar{Z}^{\alpha'}, \bar{L}^{\alpha'})\mathbf{1}_{[0,\tau]} = (\bar{Y}^\alpha, \bar{Z}^\alpha, \bar{L}^\alpha)\mathbf{1}_{[0,\tau]}$.

Continuity of the solution

- Regularity at time t of $P_t^\alpha \mapsto \bar{Y}_t^\alpha$?
- Introduction of a modulus of continuity :

$$Err_t(\eta) := \operatorname{ess\,sup}\{|\mathcal{E}^g_{t,T}[\psi^{-1}(M)] - \mathcal{E}^g_{t,T}[\psi^{-1}(M')]|, M, M' \text{ s.t. } E_t[|M-M'|^2] \leq \eta\}$$

- For any $t < T$, we get

$$|\bar{Y}_t^\alpha - \bar{Y}_t^{\alpha'}| \leq Err_t(\Delta(P_t^\alpha, P_t^{\alpha'})) + Err_t(\Delta(P_t^{\alpha'}, P_t^\alpha)),$$

where

$$\Delta : (\mu_1, \mu_2) \mapsto \frac{\mu_2 - \mu_1}{\mu_2} \mathbf{1}_{\{\mu_1 < \mu_2\}} + \frac{\mu_1 - \mu_2}{1 - \mu_1} \mathbf{1}_{\{\mu_1 > \mu_2\}}$$

- Similar properties on $\{P_t^\alpha = 0\}$ or $\{P_t^\alpha = 1\}$.
- For a Lipschitz map ψ^{-1}, stability results on classical BSDEs

$\Longrightarrow$ $\bar{Y}_t^\alpha$ is L^2-continuous with respect to P_t^α.

Convexity of the solution

- Whenever $g(.,.)$ and ψ^{-1} are convex, there exists $\hat{\alpha}$ such that $\bar{Y}^{\hat{\alpha}} = Y^{\hat{\alpha}}$

  $\Longrightarrow$ a BSDE with weak terminal condition boils down to a classical BSDE

- For any $t < T$, the solution $\bar{Y}_t^{\alpha}$ is $\mathcal{F}_t$-convex with respect to P_t^{α}.
  (need to consider the l.s.c. envelope of the solution)

- Probabilistic proof of the property.

- *"Facelift"*

  $\Longrightarrow$ if ψ deterministic, one can replace ψ^{-1} by its convex envelope

  $\Longrightarrow$ similar solutions on $[0, T)$

- In a Markovian framework, natural link with the previous PDEs.

Duality for the solution

- Suppose that g and ψ^{-1} are convex + technical conditions
- Introduce $\tilde{g}$ the Fenchel transform of g w.r.t. (y, z).
- Introduce $\tilde{\psi}^{-1}$ the Fenchel transform of ψ^{-1} w.r.t. p.
- Consider the following dual control problem :

$$\tilde{Y}_0(\ell) := \inf_{(\nu,\lambda)\in Dom(\tilde{g})} E\left[\int_0^T L_s^{\nu,\lambda}\tilde{g}(s,\nu_s,\lambda_s)ds + L_T^{\nu,\lambda}\tilde{\psi}^{-1}(\ell/L_T^{\nu,\lambda})\right]$$

$$\text{where} \qquad L_t^{\nu,\lambda} = 1 + \int_0^t L_s^{\nu,\lambda}(\nu_s ds + \lambda_s dW_s)$$

- We have the following correspondence

$$\bar{Y}_0(p) = \sup_{\ell>0}(p\ell - \tilde{Y}_0(\ell)) \qquad \text{and} \qquad \tilde{Y}_0(\ell) = \sup_{p>0}(p\ell - \bar{Y}_0(p))$$

- Standard explicit relation between the optimizers

Formal link with second order BSDEs

- Particular case of deterministic coefficients and driver independent of z
- For $\alpha \in \mathbf{L}^2$ (with $\alpha > 0$), recall that (Y^α, Z^α) is solution to the classical BSDE with driver g and terminal condition $\psi^{-1}(P^\alpha_T)$
- Denoting $B^\alpha := \int_0^\cdot \alpha_s dW_s$; $\hat{Y}^\alpha := -Y^\alpha$ and $\hat{Z}^\alpha := -Z^\alpha/\alpha$, we get

$$\hat{Y}^\alpha \quad = \quad \psi^{-1}(p + B^\alpha_T) + \int_\cdot^T -g(s, -\hat{Y}^\alpha_s)ds - \int_\cdot^T \hat{Z}^\alpha_s dB^\alpha_s \,, \quad \mathbb{P} - \text{a.s.}$$

- B^α behaves under the canonical meas. $\mathbb{P}^o$ as B under the pullback one $\mathbb{P}^\alpha$

$\Longrightarrow$ $\hat{Y}^\alpha$ under $\mathbb{P}^o$ looks like $\hat{Y}^{\mathbb{P}^\alpha}$ under $\mathbb{P}^\alpha$ where $(\hat{Y}^{\mathbb{P}^\alpha}, \hat{Z}^{\mathbb{P}^\alpha})$ solves

$$\hat{Y}^{\mathbb{P}^\alpha} \quad = \quad \psi^{-1}(p + B_T) + \int_\cdot^T -g(s, -\hat{Y}^{\mathbb{P}^\alpha}_s)ds - \int_\cdot^T \hat{Z}^{\mathbb{P}^\alpha}_s dB_s \,, \quad \mathbb{P}^\alpha - \text{a.s.}$$

- Therefore, we get : $-\bar{Y}_0 \; = \; \operatorname{ess\,sup}_\alpha \hat{Y}^\alpha_0 \; = \; \operatorname{ess\,sup}_\alpha \hat{Y}^{\mathbb{P}^\alpha}_0$.

$\Longrightarrow$ Link with 2BSDE solution but no aggregation procedure.

- Consider a time running constraint on the distribution of Y :

$$\mathbb{E}\left[\psi(Y_t)\right] \geq 0 \,, \qquad 0 \leq t \leq T \,.$$

- For any date t, the reflection is related to the law of Y_t
- Consider the BSDE dynamics

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s + K_T - K_t$$

with the previous constraint and the analogous new Skorokhod condition :

$$\int_0^T \mathbb{E}\left[\psi(Y_t)\right] dK_t = 0 \,.$$

- Dynamically non consistent problem but we derive the well posed-ness of the BSDE

BSDE with mean reflexion

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s + K_T - K_t$$
$$\mathbb{E}\left[\psi(Y_t)\right] \geq 0 \,, \qquad 0 \leq t \leq T$$
$$\int_0^T \mathbb{E}\left[\psi(Y_t)\right] dK_t = 0 \,.$$

- First observation : K must be deterministic or there is no continuous minimal solution.

  The classical penalization procedure is a priori non monotonic.

- Existence and uniqueness of the solution under the bi-Lipschitz condition :

$$c_1|x-y|\mathbf{1} \leq \; |h(x)-h(y)| \leq \; c_2|x-y|$$

- Use of a fixed point argumentation.

- The Skorokhod condition implies the minimality of the solution *(at least when the driver does not depend on Y)*

Question : BSDE with mean reflexion $\leftrightarrow$ mean field BSDE ?

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s + K_T - K_t$$
$$\mathbb{E}[\psi(Y_t)] \geq 0 \,, \qquad 0 \leq t \leq T$$
$$\int_0^T \mathbb{E}[\psi(Y_t)] dK_t = 0 \,.$$

- Can we approximate the solution of this mean-field reflected BSDE by a reflected BSDEs ?
- If $B = (B^1, \ldots, B^N)$ are independent BM, can we solve the coupled system ?

$$Y_t^{i,N} = \xi + \int_t^T g(s, Y_s^{i,N}, Z_s^{i,N})ds - \int_t^T Z_s^N \cdot dB_s + K_T^{i,N} - K_t^{i,N}$$
$$\text{with} \qquad \frac{1}{N}\sum_{i=1}^N \psi(Y_t^{i,N}) \geq 0$$

- What are the asymptotics when $N \to \infty$?

...which are on tracks :

- Addition of jumps
- Consideration of a constraint in *non linear expectation*
- Consideration of weak reflections in a dynamically consistent manner :

$$\mathbb{E}\left[\psi(Y_\tau)\right] \geq 0 , \qquad \text{for any stopping time } \tau \leq T .$$

... which should be reasonable :

- extension to a quadratic driver
- BSDE for utility maximization with quantile hedging constraint

... which seem more challenging :

- Case of coupled FBSDE for insider models
- BSDE for quantile hedging under portfolio constraints
- Consideration of one day ahead constraints : $\mathbb{E}_t[\psi(Y_{t+\delta})] \geq 0$.
- 2BSDE with weak terminal condition for robust quantile hedging
- Numerics for BSDE with weak terminal condition ?