Time inconsistent stochastic control

Dylan Possamaï, joint with Camilo Hernández

Columbia University, NY

Probability and statistics seminar, University of Southern California, Los Angeles, November 11th, 2019

Outline

Motivating example

- You receive an invitation to your first talk on a new subject a month from now.

Motivating example

- You receive an invitation to your first talk on a new subject a month from now.
- When receiving the invitation email: plenty of time left, will prepare the slides in two weeks.

Motivating example

- You receive an invitation to your first talk on a new subject a month from now.
- When receiving the invitation email: plenty of time left, will prepare the slides in two weeks.
- After two weeks: too many things to do actually, will prepare them in a week.

Motivating example

- You receive an invitation to your first talk on a new subject a month from now.
- When receiving the invitation email: plenty of time left, will prepare the slides in two weeks.
- After two weeks: too many things to do actually, will prepare them in a week.
- After three weeks: still one week left, why start now?

Motivating example

- You receive an invitation to your first talk on a new subject a month from now.
- When receiving the invitation email: plenty of time left, will prepare the slides in two weeks.
- After two weeks: too many things to do actually, will prepare them in a week.
- After three weeks: still one week left, why start now?
- Morning of the day of the talk: guess I should be starting!

Motivating example

- You receive an invitation to your first talk on a new subject a month from now.
- When receiving the invitation email: plenty of time left, will prepare the slides in two weeks.
- After two weeks: too many things to do actually, will prepare them in a week.
- After three weeks: still one week left, why start now?
- Morning of the day of the talk: guess I should be starting!

What we should take home from this: human beings have time–depending and slightly inconsistent preferences.

The basic problem

- On space $(\Omega, \mathcal{F})$, let $\mathbb{P}^\nu$ be a weak solution to the controlled SDE

$$X_t = x + \int_0^t \sigma_r(X_{r\wedge \cdot})\big(b_r(X_{r\wedge \cdot}, \nu_r)\mathrm{d}r + \mathrm{d}W_r\big),\ t \in [0, T].$$

The basic problem

- On space $(\Omega, \mathcal{F})$, let $\mathbb{P}^\nu$ be a weak solution to the controlled SDE

$$X_t = x + \int_0^t \sigma_r(X_{r\wedge\cdot})\big(b_r(X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + \mathrm{d}W_r\big),\ t \in [0, T].$$

- The reward functional

$$v(t,\nu) := J(t,t,\nu) = \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_t^T f_r(t, X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + F(t, X_{T\wedge\cdot})\bigg|\mathcal{F}_t\bigg].$$

The basic problem

- On space $(\Omega, \mathcal{F})$, let $\mathbb{P}^\nu$ be a weak solution to the controlled SDE

$$X_t = x + \int_0^t \sigma_r(X_{r\wedge\cdot})\big(b_r(X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + \mathrm{d}W_r\big),\ t \in [0, T].$$

- The reward functional

$$v(t,\nu) := J(t,t,\nu) = \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_t^T f_r(t, X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + F(t, X_{T\wedge\cdot})\bigg|\mathcal{F}_t\bigg].$$

- Control problem: because of the dependence in t, classical dynamic programming arguments fail (unless for exponential discounting of course). What to do?

The three approaches

- Dismiss the problem altogether: non–fully rational economic agents are too cumbersome.

The three approaches

- Dismiss the problem altogether: non–fully rational economic agents are too cumbersome.
- Assume precomittment: solve the maximisation problem

$$\sup_{\nu} J(0, 0, \nu),$$

and obtain an "optimal" action $\nu^\star$. However, in general $\nu^\star$ will fail to be optimal if one maximises $J(t, t, \nu)$ for $t > 0$ (Karnam, Ma, Zhang, Zhou,...)

The three approaches

- Dismiss the problem altogether: non–fully rational economic agents are too cumbersome.
- Assume precomittment: solve the maximisation problem

$$\sup_{\nu} J(0, 0, \nu),$$

and obtain an "optimal" action $\nu^\star$. However, in general $\nu^\star$ will fail to be optimal if one maximises $J(t, t, \nu)$ for $t > 0$ (Karnam, Ma, Zhang, Zhou,...)

- Take time–inconsistency seriously: consider a non–cooperative game, where the agent plays against future versions of himself, and look for sub–game perfect Nash equilibria (Barro, Czichowsky, Ekeland, Laibson, Lazrak, Pollak, Privu, Strotz, Zhou,...)

Outline

Equilibria in discrete–time

Perfectly understandable situation. If $(t_i)_{0\leq i\leq n}$ is a partition of $[0, T]$

- Given actions played by Player 0,..., and Player t_{n-2}, Player t_{n-1} faces a standard optimisation problem.

Equilibria in discrete–time

Perfectly understandable situation. If $(t_i)_{0\leq i\leq n}$ is a partition of $[0, T]$

- Given actions played by Player 0,..., and Player t_{n-2}, Player t_{n-1} faces a standard optimisation problem.

- Player t_{n-2} can solve the problem faced by Player t_{n-1} and obtains a Stackelberg equilibrium, using the optimal response of Player t_{n-1} in his own optimisation.

Equilibria in discrete–time

Perfectly understandable situation. If $(t_i)_{0\leq i\leq n}$ is a partition of $[0, T]$

- Given actions played by Player 0,..., and Player t_{n-2}, Player t_{n-1} faces a standard optimisation problem.
- Player t_{n-2} can solve the problem faced by Player t_{n-1} and obtains a Stackelberg equilibrium, using the optimal response of Player t_{n-1} in his own optimisation.
- Repeat backwardly.

Equilibria in discrete–time

Perfectly understandable situation. If $(t_i)_{0\leq i\leq n}$ is a partition of $[0,T]$

- Given actions played by Player 0,..., and Player t_{n-2}, Player t_{n-1} faces a standard optimisation problem.
- Player t_{n-2} can solve the problem faced by Player t_{n-1} and obtains a Stackelberg equilibrium, using the optimal response of Player t_{n-1} in his own optimisation.
- Repeat backwardly.
- Unfortunately, optimisation problems often lose concavity after a few iterations $\Longrightarrow$ equilibria do not exist in general.

Equilibria in continuous–time

Definition [Ekeland, Lazrak (2008)]

$\nu^\star$ is an equilibrium if for any $(t, \nu) \in [0, T) \times \mathcal{V}$

$$\liminf_{\ell \to 0} \frac{J(t, t, \nu^\star) - J(t, t, \nu^\ell)}{\ell} \geq 0,$$

where for $\ell \in [0, T - t]$, ν^ℓ is given by

$$\nu_r^\ell := \mathbf{1}_{\{t \leq r < t+\ell\}} \nu_r + \mathbf{1}_{\{t+\ell \leq r \leq T\}} \nu_r^\star.$$

Equilibria in continuous–time

Definition [Ekeland, Lazrak (2008)]

$\nu^\star$ is an equilibrium if for any $(t,\nu) \in [0,T) \times \mathcal{V}$

$$\liminf_{\ell \to 0} \frac{J(t,t,\nu^\star) - J(t,t,\nu^\ell)}{\ell} \geq 0,$$

where for $\ell \in [0, T-t]$, ν^ℓ is given by

$$\nu^\ell_r := \mathbf{1}_{\{t \leq r < t+\ell\}} \nu_r + \mathbf{1}_{\{t+\ell \leq r \leq T\}} \nu^\star_r.$$

- Definition used in most of the literature in continuous–time.

Equilibria in continuous–time

Definition [Ekeland, Lazrak (2008)]

$\nu^\star$ is an equilibrium if for any $(t,\nu) \in [0,T) \times \mathcal{V}$

$$\liminf_{\ell \to 0} \frac{J(t,t,\nu^\star) - J(t,t,\nu^\ell)}{\ell} \geq 0,$$

where for $\ell \in [0, T-t]$, ν^ℓ is given by

$$\nu^\ell_r := \mathbf{1}_{\{t \leq r < t+\ell\}}\nu_r + \mathbf{1}_{\{t+\ell \leq r \leq T\}}\nu^\star_r.$$

- Definition used in most of the literature in continuous–time.
- Not completely satisfying, when liminf is 0.

Equilibria in continuous–time

Definition [Ekeland, Lazrak (2008)]

$\nu^\star$ is an equilibrium if for any $(t,\nu) \in [0,T) \times \mathcal{V}$

$$\liminf_{\ell \to 0} \frac{J(t,t,\nu^\star) - J(t,t,\nu^\ell)}{\ell} \geq 0,$$

where for $\ell \in [0, T-t]$, ν^ℓ is given by

$$\nu^\ell_r := \mathbf{1}_{\{t \leq r < t+\ell\}} \nu_r + \mathbf{1}_{\{t+\ell \leq r \leq T\}} \nu^\star_r.$$

- Definition used in most of the literature in continuous–time.
- Not completely satisfying, when liminf is 0.
- Not a "local" property.

Equilibria in continuous–time (2)

Definition [Hernández, P. (2019)]

$\nu^\star$ is an equilibrium if for any $\varepsilon > 0$ there exists $\ell_\varepsilon > 0$ such that for any $(t, \nu, \tau) \in [0, T) \times \mathcal{V} \times \mathcal{T}_{t,t+\ell_\varepsilon}$

$$J(t, t, \nu^\star) \geq J(t, t, \nu^\tau) - \varepsilon \ell_\varepsilon.$$

If ℓ_ε can be taken independent of ε, $\nu^\star$ is called a strict equilibrium, for which

$$J(t, t, \nu^\star) \geq J(t, t, \nu^\tau).$$

Equilibria in continuous–time (2)

Definition [Hernández, P. (2019)]

$\nu^\star$ is an equilibrium if for any $\varepsilon > 0$ there exists $\ell_\varepsilon > 0$ such that for any $(t, \nu, \tau) \in [0, T) \times \mathcal{V} \times \mathcal{T}_{t,t+\ell_\varepsilon}$

$$J(t, t, \nu^\star) \geq J(t, t, \nu^\tau) - \varepsilon \ell_\varepsilon.$$

If ℓ_ε can be taken independent of ε, $\nu^\star$ is called a strict equilibrium, for which

$$J(t, t, \nu^\star) \geq J(t, t, \nu^\tau).$$

- First definition is roughly speaking the same as before: $\varepsilon-$equilibrium.

Equilibria in continuous–time (2)

Definition [Hernández, P. (2019)]

$\nu^\star$ is an equilibrium if for any $\varepsilon > 0$ there exists $\ell_\varepsilon > 0$ such that for any $(t, \nu, \tau) \in [0, T) \times \mathcal{V} \times \mathcal{T}_{t,t+\ell_\varepsilon}$

$$J(t, t, \nu^\star) \geq J(t, t, \nu^\tau) - \varepsilon \ell_\varepsilon.$$

If ℓ_ε can be taken independent of ε, $\nu^\star$ is called a strict equilibrium, for which

$$J(t, t, \nu^\star) \geq J(t, t, \nu^\tau).$$

- First definition is roughly speaking the same as before: $\varepsilon-$equilibrium.
- Strict equilibria are the ones we want, but (much) harder to prove existence (recent contribution of Huang and Zhou (2018) for CTMC).

Literature review

- Large literature in discrete–time. In continuous–time, mostly specific models, solved on a case–by–case basis.

Literature review

- Large literature in discrete–time. In continuous–time, mostly specific models, solved on a case–by–case basis.
- Ekeland, Lazrak and Pirvu introduced an extended HJB equation allowing for verification–type results: any smooth solution to the equation allows to construct an equilibrium.

Literature review

- Large literature in discrete–time. In continuous–time, mostly specific models, solved on a case–by–case basis.
- Ekeland, Lazrak and Pirvu introduced an extended HJB equation allowing for verification–type results: any smooth solution to the equation allows to construct an equilibrium.
- Extended by Björk, Khapko, and Murgoci to general Markovian diffusion models.

Literature review

- Large literature in discrete–time. In continuous–time, mostly specific models, solved on a case–by–case basis.
- Ekeland, Lazrak and Pirvu introduced an extended HJB equation allowing for verification–type results: any smooth solution to the equation allows to construct an equilibrium.
- Extended by Björk, Khapko, and Murgoci to general Markovian diffusion models.
- However, extended HJB equation only justified formally by passing to the limit in discrete–time models.

Literature review

- Large literature in discrete–time. In continuous–time, mostly specific models, solved on a case–by–case basis.
- Ekeland, Lazrak and Pirvu introduced an extended HJB equation allowing for verification–type results: any smooth solution to the equation allows to construct an equilibrium.
- Extended by Björk, Khapko, and Murgoci to general Markovian diffusion models.
- However, extended HJB equation only justified formally by passing to the limit in discrete–time models.
- Different from classical control problems, where HJB equation and optimal controls are intimately linked.

Outline

An extended DPP

Though classical DPP does not hold, can obtain

Lemma [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium, then, for any $\epsilon > 0$, $\tau \in \mathcal{T}_{0,\ell_\epsilon}$

$$v(0,\nu^\star) \leq \sup_{\nu\in\mathcal{V}} J(0,\nu),$$

$$v(0,\nu^\star) \geq \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_0^\tau f_r(0,X_{r\wedge\cdot},\nu_r)\mathrm{d}r + v(\tau,\nu^\star) + J(\tau,0,\nu^\star) - J(\tau,\tau,\nu^\star)\bigg] - \varepsilon\ell_\varepsilon.$$

If $\nu^\star$ is a strict equilibrium then

$$v(0,\nu^\star) = \sup_{\nu\in\mathcal{V}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_0^\tau f_r(0,X_{r\wedge\cdot},\nu_r)\mathrm{d}r + v(\tau,\nu^\star) + J(\tau,0,\nu^\star) - J(\tau,\tau,\nu^\star)\bigg].$$

An extended DPP

Though classical DPP does not hold, can obtain

Lemma [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium, then, for any $\epsilon > 0$, $\tau \in \mathcal{T}_{0,\ell_\epsilon}$

$$v(0,\nu^\star) \leq \sup_{\nu\in\mathcal{V}} J(0,\nu),$$

$$v(0,\nu^\star) \geq \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_0^\tau f_r(0,X_{r\wedge\cdot},\nu_r)\mathrm{d}r + v(\tau,\nu^\star) + J(\tau,0,\nu^\star) - J(\tau,\tau,\nu^\star)\bigg] - \varepsilon\ell_\varepsilon.$$

If $\nu^\star$ is a strict equilibrium then

$$v(0,\nu^\star) = \sup_{\nu\in\mathcal{V}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_0^\tau f_r(0,X_{r\wedge\cdot},\nu_r)\mathrm{d}r + v(\tau,\nu^\star) + J(\tau,0,\nu^\star) - J(\tau,\tau,\nu^\star)\bigg].$$

- Extends to arbitrary $t \in [0,T]$,

An extended DPP

Though classical DPP does not hold, can obtain

Lemma [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium, then, for any $\epsilon > 0$, $\tau \in \mathcal{T}_{0,\ell_\epsilon}$

$$v(0,\nu^\star) \leq \sup_{\nu \in \mathcal{V}} J(0,\nu),$$

$$v(0,\nu^\star) \geq \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_0^\tau f_r(0, X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + v(\tau,\nu^\star) + J(\tau,0,\nu^\star) - J(\tau,\tau,\nu^\star)\bigg] - \varepsilon\ell_\varepsilon.$$

If $\nu^\star$ is a strict equilibrium then

$$v(0,\nu^\star) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_0^\tau f_r(0, X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + v(\tau,\nu^\star) + J(\tau,0,\nu^\star) - J(\tau,\tau,\nu^\star)\bigg].$$

- Extends to arbitrary $t \in [0,T]$,
- but not to intervals of length longer than ℓ_ε.

An extended DPP (2)

Iterating the previous DPP for arbitrary partitions of $[0, T]$ with mesh smaller than ℓ_ε, and passing to the limit

Theorem [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium then

$$v(0,\nu^\star) = \sup_{\nu\in\mathcal{V}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[v(\tau,\nu^\tau) + \int_0^\tau \bigg(f(r,r,X_{r\wedge\cdot},\nu_r) - \frac{\partial F}{\partial s}(r) - \int_r^T \frac{\partial f}{\partial s}(r,u,X_{r\wedge\cdot},\nu^\star_u)\mathrm{d}u\bigg)\mathrm{d}r\bigg].$$

HJB BSDE system

Define the Hamiltonian

$$H_t(x,z) := \sup_{a\in A} h_t(t,x,z), \;\; h_t(s,x,z,a) := f_t(s,x,a) + b_t(x,a)\cdot\sigma_t(x)^\top z,$$
$$\nu_t^\star(x,z) := \mathrm{argmax}_{a\in A} h_t(t,x,z,a).$$

HJB BSDE system

Define the Hamiltonian

$$H_t(x,z) := \sup_{a\in A} h_t(t,x,z),\ \ h_t(s,x,z,a) := f_t(s,x,a) + b_t(x,a)\cdot \sigma_t(x)^\top z,$$
$$\nu_t^\star(x,z) := \mathrm{argmax}_{a\in A} h_t(t,x,z,a).$$

Extended DPP relates value of the agent at equilibrium to

$$\begin{cases} Y_t = F(T, X_{\cdot\wedge T}) + \displaystyle\int_t^T \big(H_r(X, Z_r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r, \\ Y_t^s = F(s, X_{\cdot\wedge T}) + \displaystyle\int_t^T h_r(s, X, \nu_r^\star(X_{r\wedge\cdot}, Z_r), Z_r^s)\mathrm{d}r - \int_t^T Z_r^s\cdot \mathrm{d}X_r, \end{cases}$$

where

$$\partial Y_t^s := \frac{\partial}{\partial_s} Y_t^s.$$

HJB BSDE system (2)

- For exponential discounting $f_s(t,x,a) = \mathrm{e}^{-\delta(s-t)}\tilde{f}(s,x,a)$, we have $\partial Y_t^s = -\delta Y_s \Longrightarrow$ no coupling and classical HJB BSDE.

HJB BSDE system (2)

- For exponential discounting $f_s(t,x,a) = \mathrm{e}^{-\delta(s-t)}\tilde{f}(s,x,a)$, we have $\partial Y^s_t = -\delta Y_s \Longrightarrow$ no coupling and classical HJB BSDE.
- This is the non–Markovian version of the non–local PDE system derived first by Ekeland and Lazrak (2008).

HJB BSDE system (2)

- For exponential discounting $f_s(t,x,a) = \mathrm{e}^{-\delta(s-t)}\tilde{f}(s,x,a)$, we have $\partial Y_t^s = -\delta Y_s \Longrightarrow$ no coupling and classical HJB BSDE.
- This is the non–Markovian version of the non–local PDE system derived first by Ekeland and Lazrak (2008).
- Earlier contributions argued by passing informally to the limit from discrete–time. Thanks to our extended DPP

Theorem [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium, then there exists a solution to the BSDE system, and necessarily $\nu^\star = \nu^\star(X,Z)$.

Verification theorem

Theorem [Hernández, P. (2019)]

If there exists a sufficiently integrable solution to the BSDE *system, then $\nu^\star(X,Z)$ is an equilibrium,*

$$Y_t^s = J(s,t,\nu^\star), \text{ and } Y_t = v(t,\nu^\star).$$

Verification theorem

Theorem [Hernández, P. (2019)]

If there exists a sufficiently integrable solution to the BSDE *system, then* $\nu^\star(X,Z)$ *is an equilibrium,*

$$Y_t^s = J(s,t,\nu^\star), \text{ and } Y_t = v(t,\nu^\star).$$

- Non–Markovian extension of earlier results (Ekeland and Lazrak, Björk et al., Wei et al.).

Verification theorem

Theorem [Hernández, P. (2019)]

If there exists a sufficiently integrable solution to the BSDE *system, then* $\nu^\star(X,Z)$ *is an equilibrium,*

$$Y_t^s = J(s,t,\nu^\star), \text{ and } Y_t = v(t,\nu^\star).$$

- Non–Markovian extension of earlier results (Ekeland and Lazrak, Björk et al., Wei et al.).
- The equilibrium is not necessarily strict.

Verification theorem

Theorem [Hernández, P. (2019)]

If there exists a sufficiently integrable solution to the BSDE *system, then* $\nu^\star(X,Z)$ *is an equilibrium,*

$$Y_t^s = J(s,t,\nu^\star),\ \textit{and}\ Y_t = v(t,\nu^\star).$$

- Non–Markovian extension of earlier results (Ekeland and Lazrak, Björk et al., Wei et al.).
- The equilibrium is not necessarily strict.
- Under mild conditions (Lipschitz + integrability) can prove wellposedness of the BSDE system

Extensions

- Extension to controlled volatility: 2BSDE system instead of BSDE system. Verification, and DPP still hold. But need more regularity to produce equilibria.

Extensions

- Extension to controlled volatility: 2BSDE system instead of BSDE system. Verification, and DPP still hold. But need more regularity to produce equilibria.

- More general time–inconsistency

$$J(t,t,\mathbb{M}) := \mathbb{E}^{\mathbb{P}^\nu}\left[\int_t^T f(t,r,X_{r\wedge\cdot},\nu_r)\mathrm{d}r + G(t,X_{T\wedge\cdot})\Big|\mathcal{F}_t\right] + F\Big(t,\mathbb{E}^{\mathbb{P}^\nu}[h(X_{T\wedge\cdot})|\mathcal{F}_t]\Big),$$

including mean–variance: system of 3 coupled (2)BSDEs (need to add $\mathbb{E}^{\mathbb{P}^\nu}[h(X_{T\wedge\cdot})|\mathcal{F}_t]$ as a state).

Outline

The setting

- Setting similar to Hölmstrom and Milgrom (1987). Principal hires Agent to perform a task on his behalf. Agent chooses control $\alpha \in \mathcal{A}$, and induces probability measure $\mathbb{P}^\alpha$ such that

$$X_t = x + \int_0^t \alpha_s \mathrm{d}s + \sigma W_t^\alpha,\ t \in [0, T].$$

The setting

- Setting similar to Hölmstrom and Milgrom (1987). Principal hires Agent to perform a task on his behalf. Agent chooses control $\alpha \in \mathcal{A}$, and induces probability measure $\mathbb{P}^\alpha$ such that

$$X_t = x + \int_0^t \alpha_s \mathrm{d}s + \sigma W_t^\alpha,\ t \in [0, T].$$

- Principal rewards Agent at time T with $\xi(X_{\cdot\wedge T})$. Agent's criterion is

$$J(t,\alpha,\xi) := \mathbb{E}^{\mathbb{P}^\alpha}\left[f(T-t)\xi(X_{\cdot\wedge T}) - \int_t^T f(s-t)\frac{\alpha_s^2}{2}\mathrm{d}s\right],$$

with $f(0) = 1$

The setting

• Agent looks for equilibria, so that Principal can only offer $\xi(X_{\cdot\wedge T})$ such that at least one exists. By our results, equilibria exist if and only if we have a solution to

$$Y_t = \xi(X_{\cdot\wedge T}) + \int_t^T \left(\frac{Z_r^2}{2} - \partial Y_r^r\right)\mathrm{d}r - \int_t^T Z_r \cdot \mathrm{d}X_r,$$

$$Y_t^s = f(T-s)\xi(X_{\cdot\wedge T}) + \int_t^T Z_r Z_r^s - f(r-s)\frac{Z_r^2}{2}\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r,$$

the equilibrium is $\nu_\cdot^\star := Z_\cdot$, and $J(t,\alpha^\star,\xi) = Y_t = Y_t^t$.

The BSDE system

- We can directly check that

$$Y_t^s = \frac{f(T-s)}{f(T)} Y_t^0 + f(T-s)\mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\int_t^T \left(\frac{f(r)}{f(T)} - \frac{f(r-s)}{f(T-s)}\right)\frac{Z_r^2}{2c}\mathrm{d}r\bigg|\mathcal{F}_t\right].$$

Notice that when $f(t) := \mathrm{e}^{-\delta t}$, the second term vanishes $\longrightarrow$ this is the effect due to time–inconsistency.

The BSDE system

- We can directly check that

$$Y_t^s = \frac{f(T-s)}{f(T)} Y_t^0 + f(T-s)\mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\int_t^T \left(\frac{f(r)}{f(T)} - \frac{f(r-s)}{f(T-s)}\right)\frac{Z_r^2}{2c}\mathrm{d}r\middle|\mathcal{F}_t\right].$$

Notice that when $f(t) := \mathrm{e}^{-\delta t}$, the second term vanishes $\longrightarrow$ this is the effect due to time–inconsistency.

- This implies also that

$$Z_t^s = \frac{f(T-s)}{f(T)} Z_t^0 + f(T-s)\widetilde{Z}_t^s,$$

where $\widetilde{Z}^s$ appears in the following martingale representation

$$M_t^s := \mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\int_0^T \left(\frac{f(r)}{f(T)} - \frac{f(r-s)}{f(T-s)}\right)\frac{Z_r^2}{2}\mathrm{d}r\middle|\mathcal{F}_t\right] = M_0^s + \int_0^t \widetilde{Z}_r^s \mathrm{d}X_r.$$

An upper bound for the problem

- Notice that

$$\xi(X_{\cdot\wedge T}) = \frac{Y_0}{f(T)} + \int_0^T \frac{f(r)}{f(T)} \frac{Z_r^2}{2} \mathrm{d}r + \int_0^T \sigma Z_r^0 \mathrm{d}W_r^{\alpha^\star}.$$

An upper bound for the problem

- Notice that

$$\xi(X_{\cdot\wedge T}) = \frac{Y_0}{f(T)} + \int_0^T \frac{f(r)}{f(T)} \frac{Z_r^2}{2} \mathrm{d}r + \int_0^T \sigma Z_r^0 \mathrm{d}W_r^{\alpha^\star}.$$

- Principal wants to solve

$$\begin{aligned}\sup_{\xi\in\Xi} \mathbb{E}^{\mathbb{P}^{\alpha^\star}}\left[X_T - \xi(X_{\cdot\wedge T})\right] &= x - \frac{Y_0}{f(T)} + \sup_{\xi\in\Xi} \mathbb{E}^{\mathbb{P}^{\alpha^\star}}\left[\int_0^T \left(Z_r - \frac{f(r)}{f(T)}\frac{Z_r^2}{2}\right)\mathrm{d}r\right] \\ &\leq x - \frac{Y_0}{f(T)} + \frac{1}{2}\int_0^T \frac{f(T)}{f(r)}\mathrm{d}r,\end{aligned}$$

the optimal Z being $Z_t^\star := f(T)/f(t)$

An upper bound for the problem

- Notice that

$$\xi(X_{\cdot\wedge T}) = \frac{Y_0}{f(T)} + \int_0^T \frac{f(r)}{f(T)}\frac{Z_r^2}{2}\mathrm{d}r + \int_0^T \sigma Z_r^0 \mathrm{d}W_r^{\alpha^\star}.$$

- Principal wants to solve

$$\sup_{\xi\in\Xi}\mathbb{E}^{\mathbb{P}^{\alpha^\star}}\left[X_T - \xi(X_{\cdot\wedge T})\right] = x - \frac{Y_0}{f(T)} + \sup_{\xi\in\Xi}\mathbb{E}^{\mathbb{P}^{\alpha^\star}}\left[\int_0^T\left(Z_r - \frac{f(r)}{f(T)}\frac{Z_r^2}{2}\right)\mathrm{d}r\right]$$
$$\leq x - \frac{Y_0}{f(T)} + \frac{1}{2}\int_0^T \frac{f(T)}{f(r)}\mathrm{d}r,$$

the optimal Z being $Z_t^\star := f(T)/f(t)$

- Y_0 is fixed to the reservation utility R of Agent.

Verification

- Since $Z^\star$ is deterministic, $\widetilde{Z}^s = 0$, so that

$$Z_r^0 = \frac{f(T)}{f(T-r)} Z_r^\star = \frac{f^2(T)}{f(r)f(T-r)}.$$

Verification

- Since $Z^\star$ is deterministic, $\widetilde{Z}^s = 0$, so that

$$Z^0_r = \frac{f(T)}{f(T-r)} Z^\star_r = \frac{f^2(T)}{f(r)f(T-r)}.$$

- The previous reasoning gives us the following candidate optimal contract

$$\xi^\star := \frac{R}{f(T)} + \int_0^T \frac{f(T)}{2f(r)} - \frac{f^2(T)}{f^2(r)f(T-r)} \mathrm{d}r + \int_0^T \frac{f(T)}{f(r)f(T-r)} \mathrm{d}X_r.$$

Verification

- Since $Z^\star$ is deterministic, $\widetilde{Z}^s = 0$, so that

$$Z^0_r = \frac{f(T)}{f(T-r)} Z^\star_r = \frac{f^2(T)}{f(r)f(T-r)}.$$

- The previous reasoning gives us the following candidate optimal contract

$$\xi^\star := \frac{R}{f(T)} + \int_0^T \frac{f(T)}{2f(r)} - \frac{f^2(T)}{f^2(r)f(T-r)}\mathrm{d}r + \int_0^T \frac{f(T)}{f(r)f(T-r)}\mathrm{d}X_r.$$

- Can check directly that this is an admissible contract which attains the upper bound.

Thank you for your attention!