Theoretical Problems in Credit Portfolio Modeling[2]

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Overview of Credit Portfolio Problems

- Credit portfolio risk management for general financial institutions
  - Managing credit risk at portfolio level

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  - Binomial expansion
  - SIV model

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Overview of Credit Portfolio Problems

- Credit portfolio risk management for general financial institutions
  - Managing credit risk at portfolio level
  - C-VAR
- Rating agency credit portfolio modeling
  - Binomial expansion
  - SIV model
- Credit derivative Pricing and hedging
  - Corporate
  - ABS including subprime mortgage

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General Issues and Modeling Approach

- Need a method to deal with number of names from 5 to 100 or 200. This excludes the method of modeling aggregate loss directly.

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- Need to incorporate all single name information completely for both pricing and hedging purpose
- Need to introduce correlation
- Need to have efficient computation: imagine managing a portfolio of thousands portfolio-type trades with thousands of underlying credits or collateral assets.

General Modeling Approach: Bottom-up, top-down and some in between.

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In the current copula function approach to credit, we start with creating individual credit curves independently from each other and the rest of the market. The credit curve can be described by its survival time distribution with distribution function, $F_i^*(t)$, and the density function, $f_i^*(t)$. Then we simply use a copula function $C(u_1, u_2, \ldots, u_n)$ to combine the marginal distribution functions, and *claim* that

$$F^*(t_1, t_2, \ldots, t_n) = C(F_1^*(t_1), F_2^*(t_2), \ldots, F_n^*(t_n))$$

as the *joint risk neutral distribution* of $\tau_1, \tau_2, \ldots, \tau_n$.

Current Market Practice

We use one correlation Gaussian copula function along with base correlation method.

- One correlation Gaussian copula function

$$F^*(t_1, \ldots, t_n) =$$
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- Base Correlation: To price each tranche as a difference of two equity tranches, and attach a different equity tranche correlation for different detachment point. $\max(L(t) - K_L^T, 0) - \max(L(t) - K_U^T, 0)$

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- "The formula that felled Wall Street?" The Gaussian Copula and the Material Cultures of Modeling, Donald Mackenzie and Taylor Spears, June 2012

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Correlation Parameter Compared to Moodys BET

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Correlation Parameter Compared to Moodys BET

- Use a sample transaction as an example
- It has a total name of 120 with a diversity score of 64, average rating of Baa2 and 1.35% default probability over 5-year period
- Use a normal copula function with constant correlation number to match loss distribution up to the first two moments; It is found that the correlation in normal copula function is 6.9%

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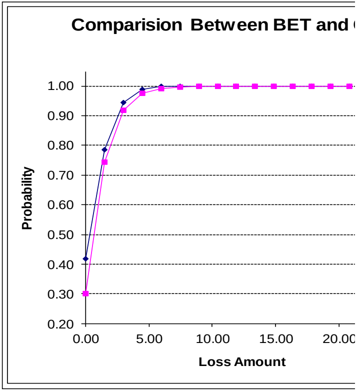
Comparison Between BET and Copula Function



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- Hedge performance for certain tranches even though not fully investigated, but is pretty poor based on our experience.
- Valuation theory from financial economics: **replication approach** and **equilibrium aapproach**.

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Replication of *n*th-to-Default CDS Contracts (I)

We introduce the following notations:

- m: the number of names in the basket.
- n: the order of the defaults specified in the swap.
- τ_i: the time of the i-th basket entity with $S_i(t), F_i(t), f_i(t)$ denoting its survival distribution, cumulative distribution, and probability density function respectively.
- $\tau_{i,m}$: the ith order statistics for m survival times, $\tau_1, \tau_2, \cdots, \tau_m$, $\tau_{1,m} \leq \tau_{2,m} \leq \ldots, \leq \tau_{m,m}$. We denote the distribution function, density function or survival function for the order statistics as $F_{n,m}(t)$, $f_{n,m}(t)$, and $S_{n,m}(t)$.
- $\tau^{(s)}_{i,m-1}$: the ith default for $m-1$ survival times, $\tau_1, \tau_2, \cdots, \tau_{s-1}, \tau_{s+1}, \cdots, \tau_m$, excluding τ_s.

Replication of nth-to-Default CDS Contracts (II)

- Two Names:

$$F_{2,2}(t) = F_1(t) + F_2(t) - F_{1,2}(t)$$

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- General Recursive Formula for order statistics (Sathe and Dixit(1990))

$$r \cdot F_{r+1,m}(t) + (m-r) \cdot F_{r,m}(t) = \sum_{i=1}^{m} F^{i}_{r,m-1}(t), \text{ for } 1 \le r \le m-1,$$

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- Every order can be expressed as the combinations of the first orders of equal or less names (Balakrishnan et al.(1992))

$$F_{n:m} = \sum_{j=m-n+1}^{m} (-1)^{j+m-n+1} \binom{j-1}{m-n} \cdot H_{1:j}(x)$$

$$H_{1:j} = \sum_{1 \leq i_1 < i_2 \ldots < i_{m-j} \leq m} F^{(i_1, i_2, \ldots i_{m-j})}_{1:j}(x)$$

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The Valuation of the *n*th-to-Default CDS Contracts

$$
\begin{aligned}
PV(\text{loss side}) &= \sum_{i=1}^{m} N(1-R_i)\int_0^T D(t)\mathbb{P}(N_t^{(i)} = n-1, \tau_i = t) \\
&= \sum_{i=1}^{m} N(1-R_i)\int_0^T D(t)\mathbb{P}(N_t = n, \tau_i = t) \\
&= \sum_{i=1}^{m} N(1-R_i)\int_0^T D(t)\mathbb{P}(\tau_{n,m} \le t, \tau_i = t) \\
&= \sum_{i=1}^{m} N(1-R_i)\int_0^T D(t)\mathbb{P}(\tau_{n,m} \le t \mid \tau_i = t) f_i(t)\, dt.
\end{aligned}
$$

For $2 \le n \le m$,

$$
(n-1)\mathbb{P}(\tau_{n,m} \le x \mid \tau_i = t) + (m-n+1)\mathbb{P}(\tau_{n-1,m} \le x \mid \tau_i = t) = \sum_{s=1}^{m} \mathbb{P}(\tau_{n-1,m-1}^{(s)} \le x \mid \tau_i = t).
$$

Merton Model (1974)

The firm's asset value V_t is specified to be GBM under the P measure. That is

$$\frac{dV(t)}{V(t)} = \mu dt + \sigma dW_t \tag{1}$$

The distance of default is defined as follows:

$$X(t) = \frac{\log\left[V(t)\right] - \log(D)}{\sigma} \tag{2}$$

$X(t)$ is a Brownian motion with a unit variance parameter and a constant drift of $m = (\mu - \frac{\sigma^2}{2})/\sigma$. Under the Merton model the default occurs at the maturity date T if

$$q(T) = P\left[X(T) \leq 0\right] = \Phi\left[B(T)\right] \tag{3}$$

where

$$B(T) = \frac{X(0) + mT}{\sqrt{T}} \tag{4}$$

Merton Model (1974)

Under risk neutral distribution we have $\mu = r$. The risk neutral default probability using the Merton model is the same as equation (3) with the change of m to $m^* = (r - \frac{\sigma^2}{2})/\sigma$. That is,

$$q^*(T) = P\left[X^*(T) \leq 0\right] = \Phi\left[B^*(T)\right] \tag{5}$$

where

$$B^*(T) = \frac{X(0) + m^* T}{\sqrt{T}} \tag{6}$$

Default Probability: Physical Measure and Risk Neutral Measure

From (3) and (5), the relationship between the risk neutral and nature or physical default probabilities is give as follows:

$$q^*(T) = \Phi\left(\Phi^{-1}\left[q(T)\right] + \frac{\mu - r}{\sigma}\sqrt{T}\right).$$

So the general relationship between the CDF functions under two measures is as follow:

$$F^*(t) = \Phi\left(\Phi^{-1}\left[F(t)\right] + \frac{\mu - r}{\sigma}\sqrt{t}\right). \tag{7}$$

Wang Transform

$$F^*(x) = \Phi\left(\Phi^{-1}\left[F(x)\right] + \lambda\right). \tag{8}$$

Intuitions of Wang Transform:

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- To make a measure change by changing its mean $\Phi^{-1}(F(X)) + \lambda$
- To use normal or other CDF to make it into a CDF function $G\left(\Phi^{-1}(F(X)) + \lambda\right)$

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Risk Neutral Distribution: Single Name

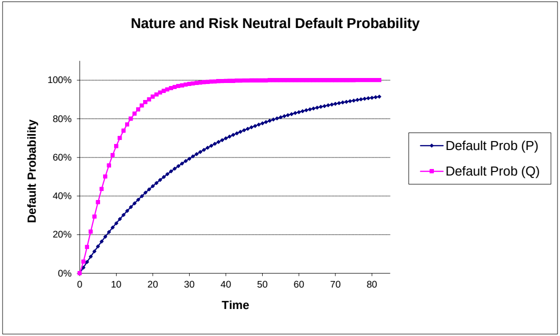

Figure: Nature and Risk Neutral Default Probabilities

Bühlmann Theorem

Bühlmann (1980) studied a risk exchange problem among a set of agents. Each agent has an exponential utility function $u(w) = -e^{-\lambda_i w}$ for $i = 1, 2, \ldots, m$ and faces a risk of potential loss X_j. In the equilibrium model, Bühlmann obtained the equilibrium pricing formula:

$$\pi(X) = E\left[\eta X\right], \eta = \frac{e^{-\lambda X}}{E\left[e^{-\lambda X}\right]}, \tag{9}$$

where $Z = \sum_{i=1}^{m} X_i$ is the aggregate risk and λ is given by

$$\lambda^{-1} = \sum_{i=1}^{m} \lambda_i^{-1}, \lambda_j > 0. \tag{10}$$

The parameter λ can be interpreted as the *risk aversion index* for a representative agent in the economy.

Measure Change in Multiasset Valuation

Multivariate equity options: how to price a multivariate equity option such as the marginal risk neutral distribution is preserved?

CDO squared pricing: How to price a CDO squared transaction such as the skew from each baby CDO tranche is matched?

A. To use single name risk neutral distributions and a copula function to form a joint risk neutral distribution

B. To have joint risk neutral transformation and then see how individual risk neutral distribution is.

Risk Neutralized with Joint Normal Distribution

Under joint Geometrical Brownian motion, the marginal distribution of the risk neutral distribution is necessarily the univariate risk neutral distribution.

$$\frac{dV_i(t)}{V_i(t)} = \mu_i dt + \sigma_i dW_i(t), E\left[W(t)W(t)^T\right] = \Sigma t$$

$$\frac{dQ}{dP} = exp\left(-\int_{t_0}^{t} \langle \lambda(s), dW(s) \rangle - \frac{1}{2}\int_{t_0}^{t} \langle \lambda(s), \lambda(s) \rangle ds\right)$$

$$\lambda(t) = [\lambda_1, \lambda_2, \ldots, \lambda_n]$$

,

$$\lambda_i = \frac{\mu_i - r}{\sigma_i}$$

Marginal Risk Neutral Distribution

$$\frac{dQ_1}{dP_1} = E\left[\frac{dQ}{dP}|W_1(t)\right] = exp\left(-\lambda_1(W_1(t) - W_1(t_0) - \frac{1}{2}\lambda_1^2(t - t_0)\right)$$

Measure Changes for Risks specified by Normal Copula

- Under physical measure each firm survival time distribution is linked to asset return distribution as follows

$$F(t) = \Phi\left(\frac{X_i(t) - \mu_i(t)}{\sigma_i\sqrt{t}}\right)$$

- $X_i(t)$ and $X_j(t)$ are joint Wiener process

We do the following:

- We first make a measure change to X_i by changing only its mean
- Then we translate the measure change from X_i to the joint measure change of survival times

Definition of Gaussian Copula for Stochastic Processes

Definition (Gaussian Copula for Stochastic Processes)

For n dimensional stochastic processes $\boldsymbol{X}(t)$, $\boldsymbol{X}(0) = \mathbf{0}$ follows a Gaussian copula with correlation coefficient Σ_ρ if there exists a standard Brownian motion vector $\boldsymbol{W}(t) = (W_1(t), W_2(t), \ldots, W_n(t))'$ whose correlation coefficient is Σ_ρ, $\boldsymbol{W}(0) = \mathbf{0}$ which makes $\boldsymbol{X}(t)$ joint distribution function to satisfy the following condition for any $t > t_0 > 0$

$$
\begin{aligned}
&P(X_1(t) < x_1, \ldots, X_n(t) < x_n \,|\, X_1(t_0) = x_{01}, \ldots, X_n(t_0) = x_{0n}) \\
=&P(W_1(t) < w_1, \ldots, W_n(t) < w_n \,|\, W_1(t_0) = w_{01}, \ldots, W_n(t_0) = w_{0n})
\end{aligned}
$$

where

$$
\begin{aligned}
w_i &= \Phi^{-1}(F_i^t(x_i)) \cdot \sqrt{t} \\
w_{0i} &= \Phi^{-1}(F_i^{t_0}(x_{0i})) \cdot \sqrt{t_0}
\end{aligned}
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Basic Properties of Gaussian Copula for Stochastic Processes

- For given time t, the Gaussian copula defined for stochastic processes is the Gaussian copula for random variable vector $\boldsymbol{X}(t)$ with the correlation coefficient is Σ_ρ.

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- If stochastic process vector $\boldsymbol{X}(t)$ satisfies the Gaussian copula condition, then $\boldsymbol{X}(t)$ is Markovian
- If stochastic process vector $\boldsymbol{X}(t)$ meets the Gaussian Copula condition, and the marginal distribution meets the following condition for any $t > t_0 > 0$, x_i, x_{0i}

$$\Phi^{-1}(F_i^t(x_i)) \cdot \sqrt{t} = \\ \Phi^{-1}(F_i^{t-t_0}(x_i - x_{0i})) \cdot \sqrt{t - t_0} + \Phi^{-1}(F_i^{t_0}(x_{0i})) \cdot \sqrt{t_0}$$

Then, $\boldsymbol{X}(t)$ has independent, and homogeneous increments.

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Esscher Measure Changes for Stochastic Processes

$$\mathbf{Y}(\mathbf{t}) = (Y_1(t_1), Y_2(t_2), \ldots, Y_n(t_n))^T$$

$$\mathbf{h} = (h_1, h_2, \ldots, h_n)$$

For the moment generation vector we use

$$\mathbf{M}(\mathbf{h}, t) = E\left[e^{\mathbf{h}^T \mathbf{Y}(t)}\right].$$

Since we have independent and stationary decrements for the joint Brownian motions we still have the following

$$\mathbf{Y}(\mathbf{t}) = [\mathbf{M}(\mathbf{h}, 1)]^t,$$

Measure Changes for Risks specified by Normal Copula(continue)

We use the positive martingale

$$\frac{e^{\mathbf{h}^T \mathbf{Y}(t)}}{[\mathbf{M}(\mathbf{h}, 1)]^t}$$

to define a new measure, the Esscher measure of parameter **h** for the stochastic processes $\mathbf{Y}(t)$. The risk neutral Esscher transform measure is the Esscher transform with parameter **h** such that, for each $i = 1, 2, \ldots, n$,

$$e^{-rt} V_i(t)$$

is a martingale.

$$e^r = \frac{\mathbf{M}(\mathbf{l}_i + \mathbf{h}^*, 1)}{\mathbf{M}(\mathbf{h}^*, 1)}, i = 1, 2, \ldots, n, \tag{11}$$

where

$$\mathbf{l}_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where the ith element is 1, and all other elements are zero

Measure Changes for Risks specified by Normal Copula (continue)

We have the following moment generating function for $\mathbf{Y(t)}$

$$\mathbf{M}(\mathbf{h}, t) = e^{t\left(\mathbf{h}^T\mu + \frac{1}{2}\mathbf{h}^T\mathbf{\Sigma}\mathbf{h}\right)}.$$

Then, we have

$$E\left[e^{\mathbf{z}^T\mathbf{Y(t)}}; h\right] = \frac{\mathbf{M}(\mathbf{z}+\mathbf{h}, t)}{\mathbf{M}(\mathbf{h}, t)} = e^{t\left(\mathbf{h}^T(\mu+\mathbf{\Sigma}\mathbf{h}) + \frac{1}{2}\mathbf{h}^T\mathbf{\Sigma}\mathbf{h}\right)}. \qquad (12)$$

Joint Risk Neutral Distribution for Survival Times

$$F^*(t_1, \ldots, t_n) =$$

$$\Phi_n\left(\Phi^{-1}[F_1(t_1)] + \lambda_1\beta_1\frac{\sigma_Z^2}{\sigma_1^2}\sqrt{t_1}, \ldots, \Phi^{-1}[F_n(t_n)] + \lambda_n\beta_n\frac{\sigma_Z^2}{\sigma_n^2}\sqrt{t_n}\right)$$

Some general comments about the results

- It solves the problem of "one period return, but multiperod survival or default"
- It introduces volatility into the framework
- It is easy to compare different portfolio
- It has a good economic foundation