Mathematical Finance Seminar @ U of Southern California

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Longevity Risk: Methods, Models, and Management

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This presentation is primarily based on a collaborations with Enrico Biffis, Fausto Gozzi, and Nan Zhu.

Financial support from the Society of Actuaries under a CAE Grant is gratefully acknowledged.

Introduction

- What is "mortality/longevity risk"?
- Is it a big deal? For whom?

Mortality Surface Models

- Motivation
- Factor Analysis of Mortality Forecasts
- Mortality Term Structure Models
- Self-consistency Condition
- Applications

Conclusion

Some folklore

- **Jeanne Louise Calment**
  - **longest** confirmed human lifespan on record
  - living to the age of 122 years, 164 days
  - died in 1997
    (she was born before the telephone was invented!)
  - Signed a **contingency contract** (similar to a reverse mortgage) at age 90 that paid 2,500 francs ($500) a month for her apartment at death. Altogether, more than 900,000 francs ($180,000)
  - Smoked till 119, loved chocolate ☺

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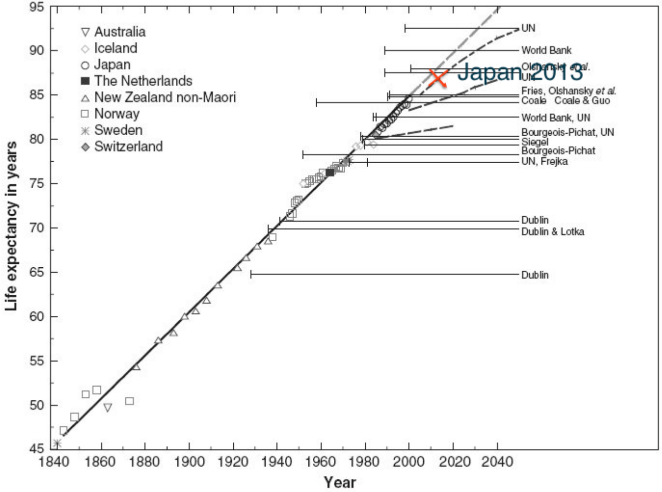
- Are we all getting older? If so, is there a limit? Is there *risk*/*uncertainty*?

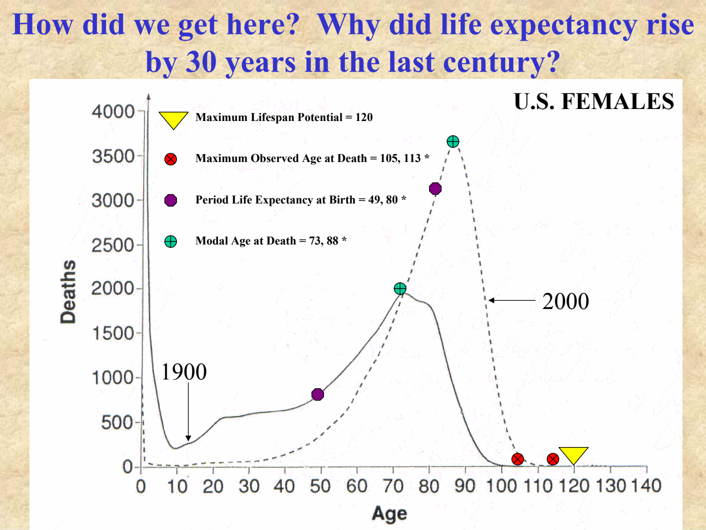
- What does this mean for individuals, corporations, and economies?

Vaupel et al. – We are getting older, and the trend has been fairly stable...



More information: Oeppen & Vaupel (2002) "Broken Limits to Life Expectancy." *Science*, 296: 1029-1031.

Olshansky et al. – ...but there may be a limit (?)



More information: Olshansky, Carnes, & Désquelles (2001) "Demography: Prospects for Human Longevity." *Science*, 291: 1491-1492.
Graph from presentation by J. Olshansky: *http://www.aaas.org/spp/rd/Olshansky.pdf*

So there is risk/uncertainty... But how grave is it?

- E.g., how will the life expectancy change over time?
  - → *Individual*: level of retirement savings
  - → *Insurance company*: (risk of) liability of deferred annuity
  - → *Economy*: Social security, retirement age

[1] Turner (2006) "Pensions, risks, and capital markets." *JRI* 73: 559-574.

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- Mitigation/Management (?)
  - Transaction design / transfer risk to individuals
    - Shift from DB to DC pensions, increase in retirement age, self-annuitization schemes, tontines, etc.
    - ? But individuals not very well equipped to take on risk post retirement...[1]
  - Securitization / mortality derivatives
    - Various transactions to transfer mortality risk to investors, at a premium (Buy-Outs, Buy-Ins, Longevity/Survivor Swaps, Longevity Bonds, etc.)
    - ? Longevity market has not lived up to expectations (reminiscent of mortgage-backed securities?) although picked up a little in recent past...

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    - ? Longevity market has not lived up to expectations (reminiscent of mortgage-backed securities?) although picked up a little in recent past...
- **Are these risks appropriately understood? Do existing models capture all the risks in an appropriate way?**

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(Insurance / Financial) Mathematics

- (We need a) **Model**: Lee-Carter model[2]

  (by far the most popular model, applied by policy makers, governmental agencies, and the broader scholarly literatures)

  $$\log\{m_{x,t}\} = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}; \kappa_t = \theta + \kappa_{t-1} + e_t$$

  ($m_{x,t}$ is the central death rate for age group x in year t)

  - Estimation typically in two step procedure using a singular value decomposition on $\{m_{x,t}\}_{x,t}$ to identify κ_t.

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- (We need) **Data**:

  - Popular: *Human Mortality Database*. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at `www.mortality.org` or `www.humanmortality.de`.

  - Other sources: CDC – NVSS or Wonder Berkeley Mortality database

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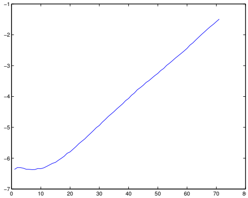
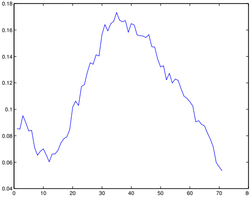
Lee-Carter

In Matlab:

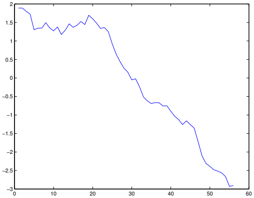
```
l_mx_female = log(mx_female);
alpha = mean(l_mx_female,2);
for t=1:T l_mx_female(:,t) = l_mx_female(:,t) - alpha * ones(X,1); end;
Sigma = cov(transpose(l_mx_female));
[eigvec,eigval] = eig(Sigma);
beta_first = eigvec(:,X);
kappa = transpose(beta_first) * l_mx_female;
deltakappa = zeros(T-1,1); for t=1:T-1  deltakappa(t) = kappa(t+1) - kappa(t); end;
theta_kap = mean(deltakappa); sigma_kap = std(deltakappa);
```



$\alpha_x \rightarrow$ Gompertz (1825): $m_x = \alpha\, e^{\beta\, x}$

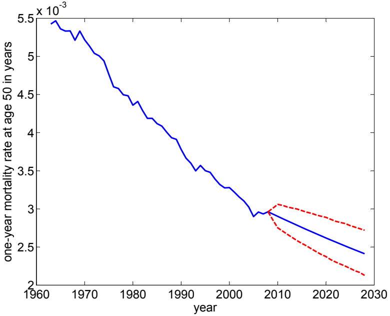


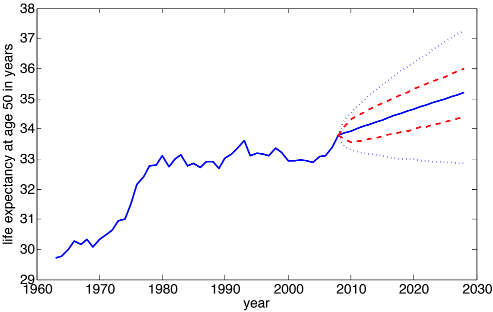
β_x



κ_t

Mortality probability and cohort life expectancy for 50-year old US female





- One year mortality probability (q_{50}) seems relatively straightforward
- Cohort life expectancy (Forward looking! Not period life expectancy as for Vaupel line!) increasing, although good bit of variation. The 95% confidence region (red dashed) seems too narrow...

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- Consider a 60 year old individual who faces *aggregate* mortality risk:
  1. Currently expected to die with probability of 0.7% next year (mortality rate). How uncertain is this appraisal? What are the chances that the rate is 0.65% or 0.75%?

  ⇒ Risk in mortality rates

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  2. Currently expected to live 25.1 more years. How uncertain is this number? What are the chances that it changes to 23.7 or 24.5 next year?

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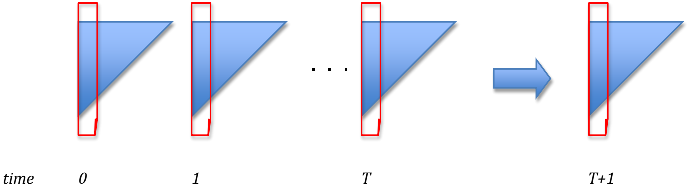
  $\Rightarrow$ Risk in mortality projections

- While the two questions are related, and the distinction does not matter in theory, it is relevant for the statistical approach
- For personal financial planning/household finance, insurers' liability risk evaluation, and government/public economics, question 2 may be more suitable

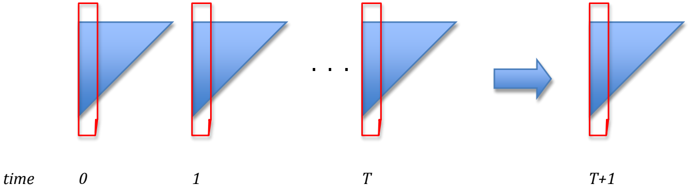


- Current stochastic mortality models focus on stochastically forecasting mortality rates (question 1)

$\rightarrow$ red in graphic

- This paper considers the risk in mortality projections (question 2)

$\rightarrow$ blue in graphic



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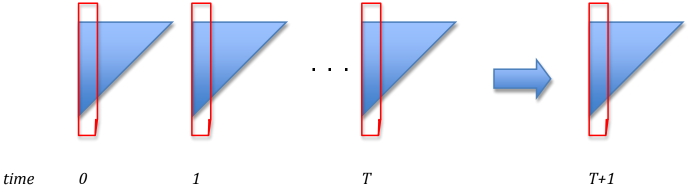

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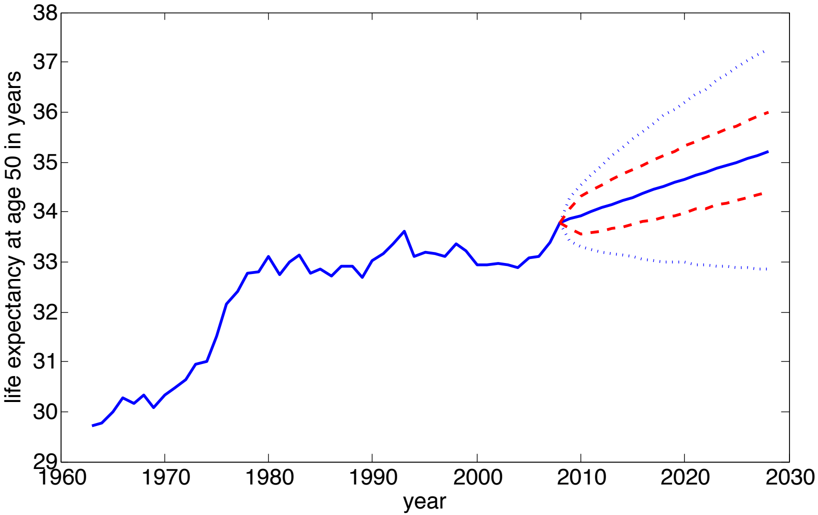
$\Rightarrow$ We want to find & estimate **tractable, *coherent* mortality models** that adequately reflect risk in mortality projections

- **Analogy in the motivation**: Term structure models
  - Forecast yield curve $p(T+1,\tau)$ based on $p(t,\tau)$ (q. 2!)
  - $p(t,\tau) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{t+\tau} r_s\, ds)]$, so (in theory) modeling r_t suffices (q. 1!)
  - Necessity to consider cross-sectional data (persistence vs. transience, etc.)
  - However, there are key differences (age, $\mathbb{P}$ vs. $\mathbb{Q}$, etc.)

Preview of Results: Forecasts of Life Expectancy



- Given: (Forward) survival probabilities $\{{}_\tau p_x(t) | (\tau, x) \in \mathcal{C}\}$
- Easier to model/work with: (Forward) force of mortality / hazard rate

$$\mu_t(\tau, x) = -\frac{\partial}{\partial \tau} \log\{{}_\tau p_x(t)\}$$

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- Assume: Time-homogeneous, Gaussian model

$$d\mu_t = (A\,\mu_t + \Lambda)\,dt + \Sigma\,dW_t$$

  - SDE on some Hilbert space $\mathcal{H}$, $A = (\partial_\tau - \partial_x)$
  - Markov, time-homogeneous (?)
    [different / much milder than Markov assumption on hazard rate!]
  - Gaussian [W fin-dim. Brownian motion] $\rightarrow$ Not necessarily positive (???)

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  $\rightarrow$ Use for now [tractability!] but also discuss non-negative models
- Define:

$$F_l\,(t_j, (\tau, x)) = -\log\left\{\frac{{}_{\tau+l}p_x(t_{j+1})}{{}_{\tau}p_x(t_{j+1})} \Big/ \frac{{}_{\tau+l+\Delta}p_{x-\Delta}(t_j)}{{}_{\tau+\Delta}p_{x-\Delta}(t_j)}\right\},\; t_{j+1} - t_j = \Delta$$

- Proposition: Vectors $\bar{F}_l(t_j) = \left(\omega(\tau, x) \times \frac{F_l(t_j,(\tau,x))}{\sqrt{t_{j+1}-t_j}}\right)_{(\tau,x)\in\tilde{\mathcal{C}}}$ iid Gaussian.

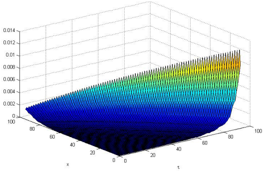
"True" mortality forecasts from market place or insurance prices

$\rightarrow$ not abundant/noisy... [key results same for "official" UK life tables]

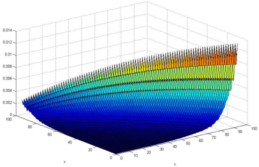
"True" mortality forecasts from market place or insurance prices
→ not abundant/noisy... [key results same for "official" UK life tables]
Raw data: (deterministic) mortality forecasts generated from rolling windows of past mortality experience [from Human Mortality Databases, www.mortality.org]

- **Regions**: England/Wales (ENW), France (FRA), Japan (JPN), United States (USA), and West Germany (FRG)
- **Genders**: male and female
- **Years**: 1956-2006 / as much as we can
- **Methods**:
  1. *Lee-Carter*
     - Weighted-least-squares algorithm
     - Ages: 0-95
  2. *CBD-Perks*
     - Basic model w/o cohort effect
     - Ages: 25-95
  3. *P-splines*
     - Fixing the degree of freedom (*df*) at 20
     - Ages: 25-95

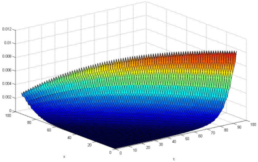
Surfaces of Eigenvectors: PC1



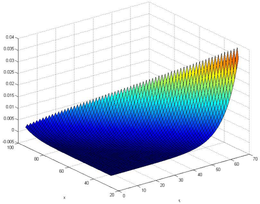
female, US, Lee-Carter



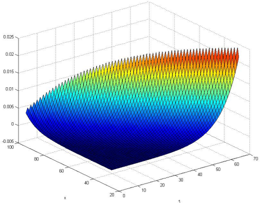
male, France, Lee-Carter



female, Ger, Lee-Carter



female, US, splines



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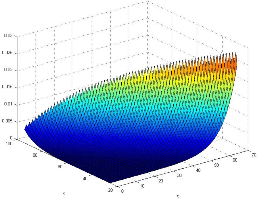
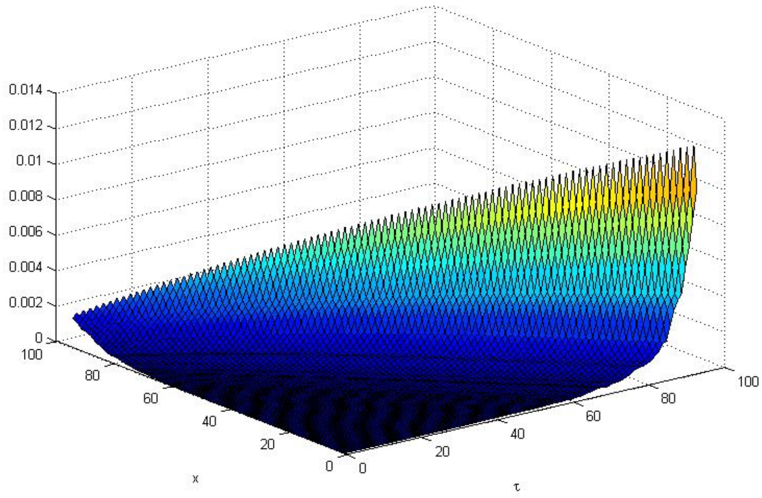


female, Ger, splines

$\rightarrow$ Explains $90 - 97\%$ of the variation!



female, US, Lee-Carter

- First two factors explain vast majority of variation in most cases (mostly $>$ 90% for first factor, first two factors $>>$ 95%)
- Very similar shapes exhibited across different countries/genders/forecasting methods (at least for the first factor)

$\Rightarrow$ PC1

- Systematic, increasing in age/term
- Forward forces of mortality for high ages in the *far future* more volatile than in the *near future*

$\rightarrow$ *slope factor*

$\Rightarrow$ PC2

- Additional effect for higher ages
- Inverse relationship between high ages in the near and in the far future

$\rightarrow$ *twist factor*

- Simple factor models by regressing $\bar{F}_l(t_j)$ on $Y_i = b_i' \bar{F}_l(t_j)$ [akin to Diebold & Li (2006, JEconometrics), Duffee (2011), etc.]

- Interpretation as forecasts:
  - Expected values for future survival probs. comprised in the current surface

  ⇒ Expectation from (stochastic) forecast should coincide with "burnt in" values
  - Cross-sectional restriction **akin to no-arbitrage condition** for interest rate term structure models

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  - $\Rightarrow$ Expectation from (stochastic) forecast should coincide with "burnt in" values
  - Cross-sectional restriction **akin to no-arbitrage condition** for interest rate term structure models
- Maths:
  - Implies martingale property that yields drift condition, i.e. $\Lambda = f(\Sigma)$
  - Prop.: μ_t allows for Gaussian finite-dimensional realization **iff**

$$\Sigma(\tau, x) = C(x+\tau) \times \exp\{M\tau\} \times N$$

[$M \in \mathbb{R}^{m\times m}$, $N \in \mathbb{R}^{m\times d}$, $C \in C([0,\infty), \mathbb{R}^m)$ – semi-parametric form]

$$\Rightarrow \mu_t(\tau, x) = \mu_0(\tau + t, x - t) + \int_0^t \Lambda(\tau + t - s, x - t + s)\, ds + C(x+\tau)\, \exp\{M\,\tau\} \underbrace{\int_0^t \exp\{M\,(t-s)\}\; N\, dW_s}_{=Z_t (\text{state process})}$$

- Identification:
  - Relationship to $\bar{F}_l$ / PCA:

$$F_l(\tau, x) \stackrel{d}{=} \mathbb{E}[F_l(\tau, x)] + \underbrace{\int_{\tau}^{\tau+1} C(x+v)\, e^{Mv}\, dv}_{=O(\tau,x)} \times \underbrace{\int_0^{\Delta} e^{M(\Delta-s)}\, N\, dW_s}_{=Z_\Delta}$$

  - Prop.: We can treat factors separately
  - Find model by regressing on principal components
    - Obtain M and N from regression on factor w/o functional assumptions on $C(x)$ [identification issues in higher dimensions – rely on examples from interest rate modeling to find convenient shapes, in particular Björk and Gombani (1999, FinStoch)]
    - Potentially use functional assumptions for capturing $C(x)$ [parametric]

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- Estimation:
  - Use Maximum Likelihood Estimation that may or may not enforce self-consistency condition
  - In addition, allow for non-systematic deviations
    - $\bar{F}^{obs}(t_j, t_{j+1}) = \bar{F}^{mod}(t_j, t_{j+1}) + \underline{\epsilon}_j$, $\underline{\epsilon}_j \sim N(0, \alpha \cdot \text{diag}\{\Sigma\})$, α small

Associated Hazard Rate

Based on our specific one-factor assumption, the **aged-dependent hazard rate** / **(spot) force of mortality** is

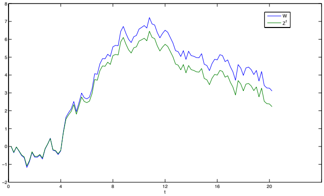
$$\mu_t(x) \;=\; \underbrace{\mu_0(t, x-t) + \int_0^t \Lambda(t-s, x-t+s)\, ds}_{\text{may use "baseline" if no } \mu_0} + C(x) \times Z_t^{(2)}$$

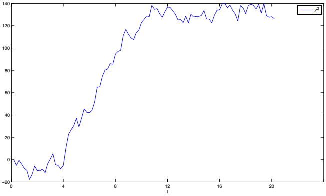
$$\overset{\text{e.g.}}{=} \;(A + B\, exp\{c\, x\}) \times (\xi_2 + Z_t^{(2)})$$

where

$$dZ_t = d \begin{pmatrix} Z_t^{(1)} \\ Z_t^{(2)} \end{pmatrix} = \begin{pmatrix} -2b & -b^2 \\ 1 & 0 \end{pmatrix} Z_t\, dt + \begin{pmatrix} 1-ab \\ a \end{pmatrix} dW_t$$

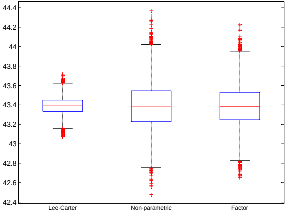
<u>Example path</u>:



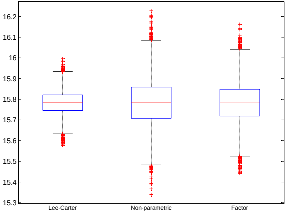


(Non-negative model: OU $\rightarrow$ Feller square root)

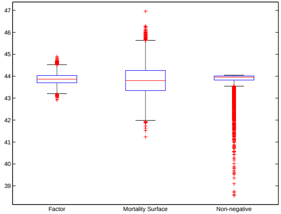
Confidence Intervals for Future LE: USA Female (After 1 yr)

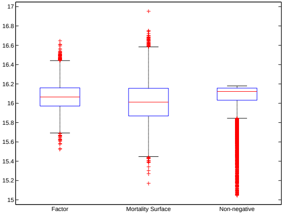


age 40



age 70





Cohort Effects

- Key difference to yield curve models: **new generations / cohort effects** (Renshaw & Haberman, 2006, IME; Cairns et al., 2009, NAAJ)

$$\log\{m_{x,t}\} = \alpha_x + \beta_x \kappa_t + \gamma_{t-x} + \varepsilon_{x,t}$$

- Deterministic models:

$$\begin{cases} \mu_t(\tau, x) &= \mu_0(\tau + t, x - t), \; x \geq t > 0, \; \tau \geq 0, \\ \mu_t(\tau, x) &= \psi_{t-x}(\tau + x), \; 0 \leq x < t \end{cases}$$

- Differential notation on $\mathcal{H}$:

$$\begin{cases} \mu_t' = A\,\mu_t - A\,B^{(0)}\,\psi_t \\ \mu_0 \in \mathcal{H}, \; \psi \in L^2_\gamma(\mathbb{R}_+) \end{cases}$$

- Stochastic extension yield SPDEs with boundary noise
- Questions:
  - Well-posedness (spaces?)
  - Finite-dimensional realizations

Introduction

- What is "mortality/longevity risk"?
- Is it a big deal? For whom?

Mortality Surface Models

- Motivation
- Factor Analysis of Mortality Forecasts
- Mortality Term Structure Models
- Self-consistency Condition
- Applications

Conclusion

Conclusion

- Having appropriate estimates for the risk in mortality projections is important
- Common approach may not be suitable to appraise risk of changes in life expectancies
- Research Directions:
  - Continuous models with cohort effects
  - Multiple populations, trade off risk?
  - Application in household finance: Annuitization decision in portfolio context/influence of systematic mortality risk
  - Asset pricing model with aggregate demographic uncertainty