Optimal Fund Menus

joint work with

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Motivation and overview

- The same firm offers many mutual funds
- Question: given the heterogeneity of investors, what is the optimal fund menu?
- We find the optimal menu when investors differ in beliefs on non-systematic risk

- The fund manager offers returns of the form

$$R(\gamma, \phi) \equiv \phi_1 \epsilon_I + \phi_2 \epsilon_{NI} - \gamma$$

- γ is the fee per dollar invested **(Linear pricing!)**
- ϕ_1, ϕ_2 are the positions in two risky assets with returns $\epsilon_I, \epsilon_{NI}$
- $\epsilon_I, \epsilon_{NI}$, are independent with means ξ, θ, and unit variances.
- θ is a subjective belief of investors, uniformly distributed on $[0, \theta_H]$.

Utility values

- Utility:

$$u(\theta, w_1) \equiv E_\theta[w_1] - \frac{a}{2}\text{var}_\theta[w_1]$$

- Given a menu **m**, investor's wealth is **(non-exclusivity!)**

$$w_1(p, \mathbf{m}) \equiv rw_0 + \int_{\mathbf{M}} R(\gamma(m), \phi(m))\, p(dm).$$

- The manager maximizes

$$v_m(\mathbf{m}) \equiv (1/\theta_H) \int_{\Theta \times \mathbf{M}} \gamma(m) p^*(dm; \theta, \mathbf{m}) d\theta$$

Revelation principle

- **Proposition 1.** W.l.o.g., we can consider the menus such that type θ invests in $(1, \phi(\theta))$.
- Denoting $\xi(\theta) \equiv (\xi, \theta)$, the optimal investment by investor is, with $\gamma = 1$, when investing in a single fund,

$$\pi(\theta, \phi) = \frac{1}{a\|\phi\|^2} \left(\phi^\top \xi(\theta) - 1\right)_+$$

  Investor's value is $v(\theta) = v(\theta, \theta)$, where

$$v(\theta, \theta') = \frac{1}{2} \left(\frac{\left(\phi(\theta')^\top \xi(\theta) - 1\right)_+}{\|\phi(\theta')\|} \right)^2$$

- The manager maximizes, over pairs (γ, ϕ), $I(\phi) \equiv (1/\theta_H) \int_\Theta \pi\left(\theta, \phi(\theta)\right) d\theta$, subject to incentive compatibility constraint that investor θ invests in $(1, \phi(\theta))$..

Relaxed problem

- **Proposition 2.** A fund loading function $\phi : \Theta \to \mathbb{R}^2$ is incentive compatible if and only if, for all θ, θ',

$$\phi(\theta')^\top \xi(\theta) - 1 - \frac{\phi(\theta)^\top \phi(\theta')}{\|\phi(\theta)\|^2} \left(\phi(\theta)^\top \xi(\theta) - 1\right)_+ \leq 0 \qquad (1)$$

- **Proposition.** Assume ϕ is incentive compatible. Then, we have

$$\pi\left(\theta, \phi(\theta)\right) = F\left(\theta, v(\theta), \dot{v}(\theta)\right) := \left(\theta \dot{v}(\theta) - 2v(\theta) + \xi \sqrt{2v(\theta) - [\dot{v}(\theta)]^2}\right)/a$$

$$\phi(\theta) \equiv \frac{1}{aF\left(\theta, v(\theta), \dot{v}(\theta)\right)} \left(\sqrt{2v(\theta) - \dot{v}(\theta)^2}, \dot{v}(\theta)\right)^\top$$

Calculus of Variations ODE

- The Euler-Lagrange ODE for the **relaxed problem** $V \equiv \sup_v \int_\Theta F\left(\theta, v(\theta), \dot{v}(\theta)\right) d\theta$ is

$$v(\theta)(1+\ddot{v}(\theta)) - [\dot{v}(\theta)]^2 = \frac{3}{2\xi}\left(2v(\theta) - [\dot{v}(\theta)]^2\right)^{\frac{3}{2}}$$

$$0 = \dot{v}(0)$$

$$\theta_H = \xi \dot{v}(\theta_H)\left(2v(\theta_H) - [\dot{v}(\theta_H)]^2\right)^{-\frac{1}{2}}.$$

- **Theorem.** There exists a unique solution v^* to the ODE, it is strictly increasing, strictly convex, it attains the supremum in the relaxed problem, and it gives rise to the solution of the original problem.

Solution

- **Properties:**
  - No investor is excluded, $F(\cdots) > 0$.
  - Manager is better off with higher ξ and θ_H
  - $\phi_2^*(0) = 0$: the manager offers an index fund.
  - Let $f(\theta)$ denote the first best optimal funds. We have $\phi_2^*/\phi_1^* = f_2/f_1$ at $\theta = 0$ and $\theta = \theta_H$.
  - Otherwise, $\Delta = f_2/f_1 - \phi_2^*/\phi_1^* > 0$ and it is inverse $U-$shaped - **closet indexing**.
  - There exist θ_1, θ_2 such that the optimal exposure to the index is lower than in the first best for $\theta \leq \theta_1$ and higher otherwise, the optimal exposure to the non index asset is lower than in the first best for $\theta \leq \theta_2$ and higher than in the first best for $\theta > \theta_2$.
  - We have that $\theta_1 < \theta_2$. Therefore, low types are under-invested in both assets, high types are over-invested in both assets, and intermediate types in $[\theta_1, \theta_2]$ are overinvested in the index and under-invested in the non index asset.
  - There exists an intermediate type $\bar{\theta} \in (\theta_1, \theta_2)$ such that investors below $\bar{\theta}$ have lower utility than in the first best while investors above have higher utility than in the first best.

Exogenous index fund

- Investors can invest in an outside index fund (ϵ_I, γ_I).
- Revelation principle still holds.
- **Lemma.** If

$$\gamma_I \geq \gamma_I^* \equiv \xi - \sqrt{2v^*(0)}$$

then, the optimal fund menu is as before.

Additional IC constraint

- IC now requires also

$$2v(\theta) \geq (\xi - \gamma_I)^2 + [\dot{v}(\theta)]^2$$

- Lagrangian:

$$\int_\Theta H^\lambda(\theta, v(\theta), \dot{v}(\theta))\, d\theta \equiv \int_\Theta \{F(\theta, v(\theta), \dot{v}(\theta)) + \lambda(\theta) c(v(\theta), \dot{v}(\theta))\}\, d\theta$$

where

$$c(v(\theta), \dot{v}(\theta)) \equiv 2v(\theta) - [\dot{v}(\theta)]^2 - (\xi - \gamma_I)^2$$

- **Lemma.** Denote by $\mathcal{C}$ the set of points where the function λ is continuous. If

$$\left(H^\lambda_{v(\theta)} - \tfrac{d}{d\theta} H^\lambda_{\dot{v}(\theta)}\right)(\theta, v(\theta), \dot{v}(\theta)) = 0, \qquad \theta \in \mathcal{C}, \tag{2}$$

$$H^\lambda_{\dot{v}(\theta)}(\theta, v(\theta), \dot{v}(\theta)) = 0, \qquad \theta \in \{0, \theta_H\}, \tag{3}$$

$$\lambda(\theta) c(v(\theta), \dot{v}(\theta)) = 0, \qquad \theta \in \Theta, \tag{4}$$

and $H^\lambda_{\dot{v}(\theta)}(\theta, v(\theta), \dot{v}(\theta))$ is continuous (equivalent to $\dot{v}(\theta) = 0$ for θ not in $\mathcal{C}$), then v attains the supremum in the relaxed problem.

Solution

- a) Assume that $\gamma_I < \xi/3$, then the function

$$v^*(\theta) \equiv \frac{1}{2}(\xi - \gamma_I)^2 + \frac{1}{2}\left(\theta - \frac{\theta_H}{3}\right)_+^2 \tag{5}$$

  attains the supremum in the relaxed problem.

  b) Assume that $\gamma_I \in [\xi/3, \gamma_I^*)$ and denote by $(w, \theta^*) \in C_p^2(\Theta; \mathbb{R}) \times \Theta$ the unique solution to the free boundary problem defined by

$$0 = w(\theta)(1 + \ddot{w}(\theta)) - [\dot{w}(\theta)]^2 - \frac{3}{2\xi}\left(2w(\theta) - [\dot{w}(\theta)]^2\right)^{\frac{3}{2}}, \qquad \theta \in \Theta, \tag{6}$$

  subject to the boundary conditions

$$0 = \dot{w}(\theta^*) = w(\theta^*) - \frac{1}{2}(\xi - \gamma_I)^2, \tag{7}$$

$$= \theta_H - \xi \dot{w}(\theta_H)\left(2w(\theta_H) - [\dot{w}(\theta_H)]^2\right)^{-\frac{1}{2}}. \tag{8}$$

  Then the function

$$v^*(\theta) \equiv \frac{1}{2}(\xi - \gamma_I)^2 + \mathbf{1}_{\{\theta > \theta^*\}}\left(w(\theta) - \frac{1}{2}(\xi - \gamma_I)^2\right) \tag{9}$$

  attains the supremum in the relaxed problem.

Properties

- The manager optimally offers the index to all investors below a certain cutoff type.
- When $\gamma_I < \xi/3$ the constraint binds for all types and the optimal menu provides the utilities to all players that are the same as in the case in which the manager offers two funds given by $(\gamma_I, \mathbf{e}_1)$ and $(\gamma_{NI}, \mathbf{e}_2)$, with $\gamma_{NI} = \theta_H/3$.

Future research?

- A screening problem with
  - multiple goods
  - flexible quantities
  - buyers can mix contracts
  - unobserved preferences on some of the goods
  - linear pricing
  - the seller can offer the products in bundles

Proof of Proposition 1.

Lemma. It is sufficient for investor θ to invest only in two funds.
This is because the optimization problem of investor θ can be written as

$$v_i(\theta, \mathbf{m}) = \sup_{x \in \mathbb{R}^2} \sup_{p \in \mu_+^x(\mathbf{M})} \left\{ x_1 \xi_1 + x_2 \theta - \frac{1}{2} a \|x\|^2 - \int_{\mathbf{M}} \gamma(m) p(dm) \right\} \quad (10)$$

where

$$\mu_+^x(\mathbf{M}) = \left\{ p \in \mu_+(\mathbf{M}) : \int_{\mathbf{M}} \phi(m) p(dm) = x \right\}. \quad (11)$$

Then, the result follows from standard deterministic optimization results.

Proof of Proposition 1.

Suppose investor of type $\theta \in \Theta$ allocates money to a pair of funds $(m_1(\theta), m_2(\theta))$. We need to choose the fund loading vector $\phi(\theta)$ so that

$$\sum_{k=1}^{2} \gamma_0(m_k(\theta))p_k(\theta) = \pi(\theta, \phi(\theta)), \tag{12}$$

$$\frac{a}{2}\left\| \sum_{k=1}^{2} p_k(\theta)\phi_0(m_k(\theta)) \right\|^2 = v(\theta) = \frac{1}{2a\|\phi(\theta)\|^2}\left(\phi(\theta)^\top \xi(\theta) - 1\right), \tag{13}$$

The solution is

$$\phi(\theta) = \frac{p_1(\theta)\phi_0(m_1(\theta)) + p_2(\theta)\phi_0(m_2(\theta))}{p_1(\theta)\gamma_0(m_1(\theta)) + p_2(\theta)\gamma_0(m_2(\theta))}. \tag{14}$$

ODE

Lemma. Assume that $v^* \in C^2(\Theta; \mathbb{R})$ is a solution to the boundary value problem. Then, v^* is optimal for the relaxed problem.
Proof: Let v be another feasible function. It can be shown that F is concave, so that

$$\int_\Theta \Big(F\left(x, v(\theta), \dot{v}(\theta)\right) - F\left(x, v^*(\theta), \dot{v}^*(\theta)\right) \Big) d\theta \leq \Delta(v, v^*) \tag{15}$$

$$\equiv \int_\Theta \Big(\left(v(\theta) - v^*(\theta)\right) F^*_{v^*(\theta)}(\theta) + \left(\dot{v}(\theta) - \dot{v}^*(\theta)\right) F^*_{\dot{v}^*(\theta)}(\theta) \Big) d\theta \tag{16}$$

Integration by parts shows that

$$\Delta(v, v^*) = \left(\left(v - v^*\right)(\theta) F^*_p(\theta) \right) \Big|_{\theta=0}^{\theta_H} + \int_\Theta \left(v - v^*\right)(\theta) \left(F^*_v(\theta) - \frac{d}{d\theta} F^*_p(\theta) \right) d\theta \tag{17}$$

$$= \left(v(\theta_H) - v^*(\theta_H)\right) F^*_p(\theta_H) - \left(v(0) - v^*(0)\right) F^*_p(0) = 0 \tag{18}$$

where the last two equalities follow from the fact that v^* solves the ODE and the boundary conditions.

Thank you for your attention!

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