Systemic Influences on Optimal Equity-Credit Investment

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Overview

- **Goal:** analyze how systemic risk affects optimal investment
- **Risk types:** stock prices are affected by
  - market risk: day-to-day fluctuations in stock prices
  - default risk: company's potential failure to pay its obligations
- **Risk dependencies:**
  - structural interaction of market and default risk
  - systemic dependencies between different stocks
- **Theoretical study:**
  - model with different investment possibilities taking these risk types and dependencies into account
  - dynamic investment problem $\Rightarrow$ explicit solution
- **Empirical analysis:**
  - new calibration procedure
  - systemic influence $\Rightarrow$ importance in risk management

An investment model with risk dependencies

Stock prices

The stock price dynamics of company $i = 1, \ldots, M$ is given by

$$\mathrm{d}S_i(t) = S_i(t-)\big(\mu_i \,\mathrm{d}t + \mathbf{\Sigma}_i \,\mathrm{d}\mathbf{W}(t) - \big(\mathrm{d}H_i(t) - h_i(t, \mathbf{S}(t), \mathbf{F}(t))\,\mathrm{d}t\big)\big),$$
$$S_i(0) > 0,$$

which has the components

- instantaneous drift $\mu_i \,\mathrm{d}t$ for $\mu_i \in \mathbb{R}$,
- market risk $\mathbf{\Sigma}_i \,\mathrm{d}\mathbf{W}(t)$ for a d-dimensional Brownian motion $\mathbf{W}(t)$ and $\mathbf{\Sigma}_i \in \mathbb{R}^d$,
- default risk $\mathrm{d}H_i(t)$ for the default indicator process $H_i(t)$, which is zero before the default time of the i^{th} company and jumps at one at the default time,
- compensated instantaneous default risk $h_i(t, \mathbf{S}(t), \mathbf{F}(t))\,\mathrm{d}t$.

On the default risk:

- Intuitively, the probability that company i defaults over an infinitesimal time interval $[t, t + \mathrm{d}t]$ equals

$$\mathbb{P}[H_i(t) = 0 \text{ and } H_i(t + \mathrm{d}t) = 1] = h_i(t, \mathbf{S}(t), \mathbf{F}(t))\,\mathrm{d}t.$$

- The default intensity $h_i(t, \mathbf{S}(t), \mathbf{F}(t))$ of company i may change if
  1. its own stock price $S_i(t)$ changes,
  2. the stock price $S_j(t)$ of another company $j \neq i$ changes,
  3. the additional factors $\mathbf{F}(t)$ change.

We assume that, for every $n \in \mathbb{N}$,

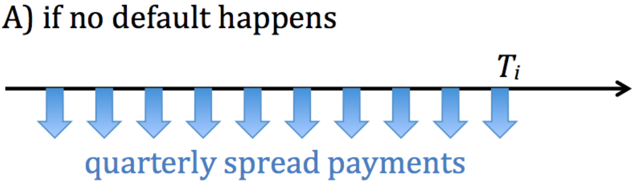
> $h_i(t, s_1, \ldots, s_M, f_1, \ldots, f_K)$ are nonnegative, uniformly Lipschitz continuous and bounded (bound can depend on n) for $t \geq 0$, $s_i \geq 1/n$, $s_j \geq 0$ with $j \neq i$ and $f_\ell \in \mathbb{R}$.

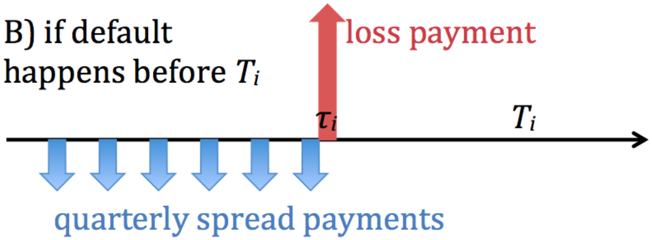
Lemma

Under this assumption, the model is well defined and there is a unique solution to the stochastic differential equation of $\mathbf{S}(t)$.

CDS prices

- For each company $i = 1, \ldots, M$, we consider a corresponding CDS (credit default swap) with maturity T_i.
- If the default of the i^{th} company happens at $\tau_i < T_i$, the protection buyer receives a payment of L_i per unit notional i^{th} CDS from the protection seller.
- The protection buyer pays the spread premium ν_i to the protection seller until the earlier of default time or maturity.





- The cumulative dividend process of the unit notional i^{th} CDS received by the protection buyer is given by

$$D_i(t) = \underbrace{L_i H_i(t)}_{\text{payment at default time}} - \underbrace{\nu_i \int_0^t (1 - H_i(s))\, ds}_{\text{cumulative payments before default occurs}} .$$

Pricing measure:

- As for any traded derivative, the CDS price is equal to the expected discounted payoff under a risk-neutral probability measure $\mathbb{Q}$.
- It is important to distinguish $\mathbb{Q}$ from the subjective probability measure $\mathbb{P}$ of the investor.
- Under suitable assumptions, one can show that the price of the i^{th} CDS is Markov in $\mathbf{S}(t)$ and given by

$$C_i(t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^{T_i} \mathrm{e}^{-r(u-t)} \mathrm{d}D_i(u) \middle| \mathbf{S}(t) \right].$$

$\Longrightarrow$ We can write $C_i(t) = \Phi_i(t, \boldsymbol{S}(t))$ for some function Φ_i.

Default intensity under $\mathbb{Q}$:

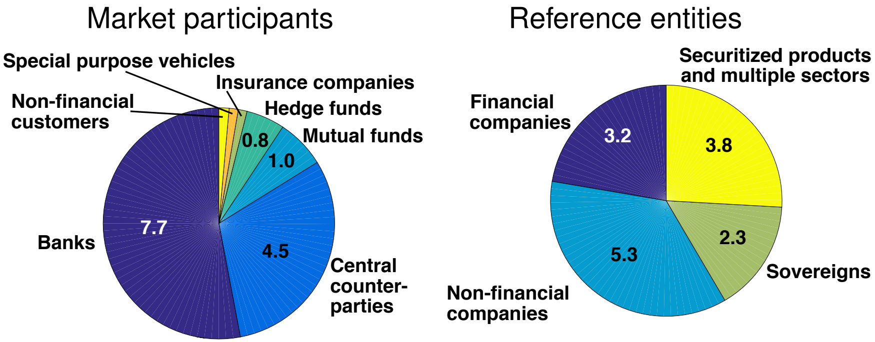
- We denote by $\lambda_i(t, \boldsymbol{S}(t))$ the default intensity of stock i under the risk-neutral probability measure $\mathbb{Q}$,

$$\underbrace{\lambda_i(t, \boldsymbol{S}(t))}_{\text{risk-neutral}} \neq \underbrace{h_i(t, \boldsymbol{S}(t), \boldsymbol{F}(t))}_{\text{subjective}} \quad \text{because} \quad \mathbb{Q} \neq \mathbb{P}.$$

Untriggered CDS:

- We say that the i^{th} CDS is untriggered at t if the default of the i^{th} company has not yet occurred and $t \leq T_i$.
- We denote by $\mathbb{M}(t) \subseteq \{1, \ldots, M\}$ the set of untriggered CDSs at time t.

CDS market



in trillion USD notional amounts for the first half of 2015

Source: Bank for International Settlements

Dynamic investment problem

- Initial wealth: The investor starts with a positive wealth $V(0)$ at time zero.
- Investment possibilities: Investments are possible in bank account, M stocks and M CDSs on the same companies during $[0, T)$.
- Stock investment: As usual in optimal investment problems, we denote by $\pi_i(t)$ the proportion of total wealth invested in the i^{th} stock.
- CDS investment: Conditions of CDS contract depend on the entering date, but standardized maturity date for all contracts entered in the same quarter.

Parametrizing CDS strategy:

- We can denote by $\hat{\psi}_i(t)$ the net position in shares of the i^{th} CDS at time t for a given maturity T_i. We define

$$\psi_i(t) = \frac{\hat{\psi}_i(t)}{V(t)} = \frac{\text{number of shares of the } i^{\text{th}} \text{ CDS}}{\text{total wealth}}.$$

  Note that we do **not** multiply by the CDS price, hence $\psi_i(t)$ is **not** the proportion of wealth invested in the i^{th} CDS.

- Differently from the stock, the CDS price can be zero even before the default event has occurred.
  $\Longrightarrow$ Important to know units and not only dollar amount.

- Assume neither intermediate consumption nor capital income (self-financing condition) $\Longrightarrow$ position in the bank account is determined as residual.

Optimization problem:

- Our goal is to find the optimal strategy, which maximizes the expected utility from terminal wealth

$$\mathbb{E}^{\mathbb{P}}\left[U\left(V^{\pi,\psi}(T)\right)\right]$$

  over all admissible (π, ψ), where $T \in (0, \min(T_1, \ldots, T_M))$.

- We define a strategy to be admissible if it has positive wealth and satisfies certain technical conditions.
- We consider logarithmic utility, i.e., $U(v) = \log(v)$ because
  - it allows exemplifying typical systemic effects,
  - otherwise, the problem is not tractable,
  - there is empirical evidence (Gordon, Paradis and Rorke, *American Economic Review* 1972) that wealthy investors, such as the highly specialized market players in the CDS market, maximize expected logarithmic utility.

The main result

Preparation

Define a stochastic matrix $\mathbf{\Theta} = (\Theta_{n,i})_{n,i\in\mathbb{M}(t)}$ with entries

$$\Theta_{n,i} = \text{“total derivative of } i^{\text{th}} \text{ CDS price with respect to default and market risk of the } n^{\text{th}} \text{ stock"}$$

$$= \begin{cases} L_i - C_i(t) + \frac{\partial\Phi_i}{\partial s_i}(t,\boldsymbol{S}(t))S_i(t) & \text{if } i = n \\ \Phi_i\big(t,\boldsymbol{S}^{(n)}(t)\big) - C_i(t) + \frac{\partial\Phi_i}{\partial s_n}(t,\boldsymbol{S}(t))S_n(t) & \text{if } i \neq n. \end{cases}$$

In the above expression, $\boldsymbol{S}^{(n)}(t)$ equals $\boldsymbol{S}(t)$ except for the n^{th} component of $\boldsymbol{S}^{(n)}(t)$, which equals zero. Hence,

$$\Phi_i\big(t,\boldsymbol{S}^{(n)}(t)\big) = \text{price of } i^{\text{th}} \text{ CDS when } n^{\text{th}} \text{ stock defaults.}$$

We assume that

the stochastic matrix $\mathbf{\Theta}$ has full rank almost everywhere.

Lemma

The following are equivalent:

1. *The matrix $\mathbf{\Theta}$ has full rank almost everywhere.*
2. *The risk-neutral default intensity is unique.*

$\Longrightarrow$ CDS price dynamics are uniquely determined by market.

Example: Assume all default intensities λ_n are constant
$\Longrightarrow$ matrix $\mathbf{\Theta}$ is diagonal with strictly positive entries $L_n - C_n(t)$
$\Longrightarrow$ the assumption on full rank of $\mathbf{\Theta}$ trivially holds.

The optimal investment strategy

Theorem

The optimal investment strategy in untriggered CDSs and stock is given by

$$\begin{aligned}\psi(t) &= \boldsymbol{\Theta}^{-1}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\top}\right)^{-1}(\mu_n - r + h_n - \lambda_n)_{n\in\mathbb{M}(t)} \\ &\quad + \boldsymbol{\Theta}^{-1}\left(\frac{h_n - \lambda_n}{\lambda_n}\right)_{n\in\mathbb{M}(t)}, \\ \boldsymbol{\pi}(t) &= \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\top}\right)^{-1}(\mu_n - r + h_n - \lambda_n)_{n\in\mathbb{M}(t)} \\ &\quad - \left(\sum_{i\in\mathbb{M}(t)} \psi_i(t) S_n(t) \frac{\partial \Phi_i}{\partial s_n}(t, \mathbf{S}(t))\right)_{n\in\mathbb{M}(t)}.\end{aligned}$$

Analysis of optimal stock investment:

In the classical Merton equity selection problem, one has

$$\pi(t) = \frac{\mu - r}{\sigma^2}$$

$\Downarrow$ several stocks

$$\boldsymbol{\pi}(t) = \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\right)^{-1}(\mu_n - r)_{n=1,\dots,M}$$

$\Downarrow$ default risk

$$\boldsymbol{\pi}(t) = \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\right)^{-1}(\mu_n - r + h_n - \lambda_n)_{n\in\mathbb{M}(t)}$$

$-$ an additional vector

Analysis of additional vector:

n^{th} component of additional vector

$$= \sum_{i \in \mathbb{M}(t)} \psi_i(t) S_n(t) \frac{\partial \Phi_i}{\partial s_n}(t, \mathbf{S}(t))$$

$=$ sensitivity of the total CDS exposure to the market risk of company n

$\Longrightarrow$ Investor uses stocks also to hedge the market risk of the CDS position.

The maximal expected utility

Theorem

The maximal expected utility is given by

$$\begin{aligned}\mathbb{E}\big[U\big(V^{\pi,\psi}(T)\big)\big] &= \log(V(0)) + rT \\ &\quad + \frac{1}{2}\mathbb{E}\bigg[\int_0^T (\mu_n - r + h_n - \lambda_n)^\top_{n\in\mathbb{M}(t)} \big(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\big)^{-1} \\ &\qquad\qquad (\mu_n - r + h_n - \lambda_n)_{n\in\mathbb{M}(t)}\,\mathrm{d}t\bigg] \\ &\quad + \sum_{n=1}^{M}\mathbb{E}\bigg[\int_0^{\tau_n\wedge T}\bigg(-\log\Big(\frac{\lambda_n}{h_n}\Big) + \frac{\lambda_n}{h_n} - 1\bigg)h_n\,\mathrm{d}t\bigg].\end{aligned}$$

Analysis of maximal expected utility:

In the classical Merton equity selection problem, one has

$$\mathbb{E}\big[U\big(V^{\pi}(T)\big)\big] = \frac{1}{2}\underbrace{\mathbb{E}\Big[\int_0^T \frac{(\mu - r)^2}{\sigma^2}\,\mathrm{d}t\Big]} + \log(V(0)) + rT.$$

$\Downarrow$ several stocks

$$\mathbb{E}\Big[\int_0^T (\mu_n - r)_{n=1,\dots,M}^{\top}\big(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\top}\big)^{-1}(\mu_n - r)_{n=1,\dots,M}\,\mathrm{d}t\Big]$$

$\Downarrow$ default risk

$$\mathbb{E}\Big[\int_0^T (\mu_n - r + h_n - \lambda_n)_{n\in\mathbb{M}(t)}^{\top}\big(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\top}\big)^{-1}(\mu_n - r + h_n - \lambda_n)_{n\in\mathbb{M}(t)}\,\mathrm{d}t\Big]$$

+ an additional term

Analysis of additional term:

$$\sum_{n=1}^{M} \mathbb{E}\left[\int_0^{\tau_n \wedge T} \left(-\log\left(\frac{\lambda_n}{h_n}\right) + \frac{\lambda_n}{h_n} - 1 \right) h_n \, \mathrm{d}t \right]$$

Using $-\log(x) + x - 1 \geq 0$ for all $x > 0$ with equality for $x = 1$, we see for $x = \lambda_n / h_n$ that

- no utility gain if subjective view = market view
- always utility gains if subjective view $\neq$ market view
  - if $x = \lambda_n / h_n < 1$, investor buys default protection believing that market is underpricing default risk of company n
  - if $x = \lambda_n / h_n > 1$, investor sells default protection believing that market is overpricing default risk of company n

Empirical analysis

Calibration procedures

- We develop an empirical analysis to identify
  - why systemic influences arise,
  - how they propagate to drive optimal investment decisions.
- Companies are from the Dow Jones Industrial Average 30 (DJIA) as of December 31, 2007, and the experimental period is the year 2008.
- The calibration of the subjective default intensity

$$h_n = \mathrm{e}^{c_0 + c_1 w_1^n + c_2 w_2^n + c_3 w_3 + c_4 w_4}$$

  is based on the model by Duffie, Saita and Wang (2007) with constants $c_0, \ldots, c_4$ and four main covariates:
  1. w_1^n: the company's distance to default,
  2. w_2^n: the company's trailing one-year stock return,
  3. w_3: the three-month Treasury bill rate,
  4. w_4: the trailing one-year return on the DJIA.

Calibration of risk-neutral default intensity

- We first specify a parametric model

$$\lambda_n(t, \mathbf{S}(t)) = \alpha_n + \underbrace{\beta_n / S_n(t)}_{\text{idiosyncratic}} + \underbrace{\gamma_n \#\{i : S_i(t) = 0\}}_{\text{systemic}},$$

  where α_n, β_n and γ_n are constants, and $\#\{i : S_i(t) = 0\}$ is the number of companies that have defaulted by time t.
- We estimate α_n, β_n and γ_n by calculating CDS prices implied by our default intensity specification and matching them to empirically observed data.

$$\alpha_n, \beta_n, \gamma_n \Rightarrow \lambda_n(t, \mathbf{S}(t)) \Rightarrow \text{dynamics of } S_n \Rightarrow n^{\text{th}} \text{ CDS price}$$

$$\alpha_i, \beta_i, \gamma_i \Rightarrow \lambda_i(t, \mathbf{S}(t)) \Rightarrow \text{dynamics of } S_i \Rightarrow i^{\text{th}} \text{ CDS price}$$

Problem: all 90 parameters affect all CDS prices, coupled minimization problem: computationally intractable.

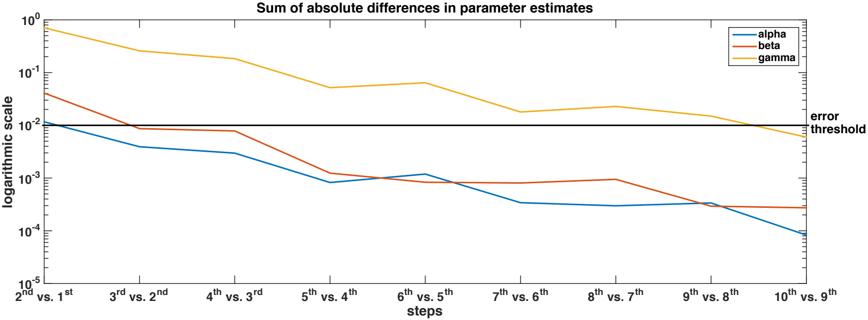
More suitable is the iterative procedure:

1st Step
- We assume the absence of systemic risk (all γ_n are zero) and estimate α_n and β_n.
- $\alpha_n, \beta_n \Rightarrow \lambda_n(t, S_n(t)) \Rightarrow S_n \Rightarrow n^{\text{th}}$ CDS price
  decoupled minimization problems: tractable.

2nd Step
- For each $n = 1, \ldots, 30$, we solve an optimization problem over three variables α_n, β_n and γ_n.
- For $i \neq n$, we use the values of α_i and β_i estimated in the first step, and set $\gamma_i = 0$.
- $\alpha_n, \beta_n, \gamma_n \Rightarrow \lambda_n(t, \mathbf{S}(t)) \Rightarrow S_n \Rightarrow n^{\text{th}}$ CDS price
  α_i, β_i from first step and $\gamma_i = 0$ for $i \neq n$,
  decoupled minimization problems: tractable.

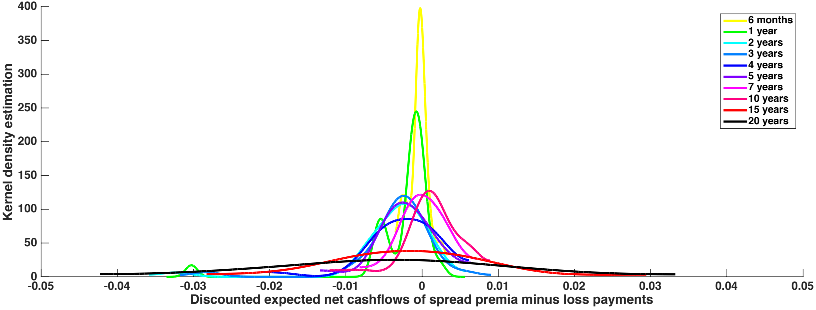
K^{th} Step
- Proceed like in 2nd Step, but for $i \neq n$, we use the values of α_i, $\beta_i$and γ_i from Step $K - 1$.

Convergence of iterative procedure:



- We first sum, over all companies, the absolute differences in parameter estimates from two consecutive steps of the calibration procedure, for each trading day in Dec. 2007.
- We then take the average over the trading days.

Error distribution across different maturities:



- As expected, the errors are smaller for shorter maturities.
- The symmetric distributions indicate the absence of any systematic pricing bias across different maturities.

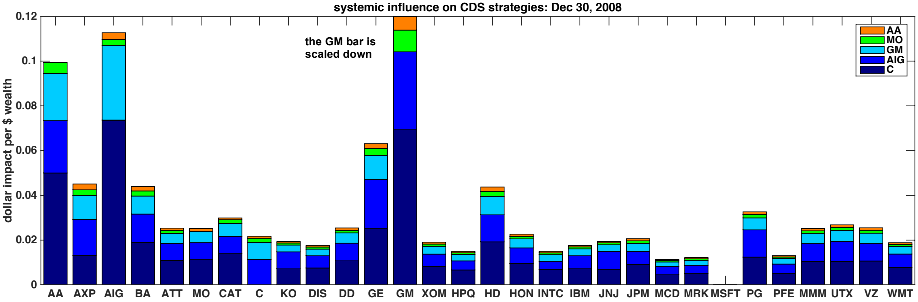
Experimental results

- Our main goal is to quantify the impact of systemic risk influences and monitor their evolution over time.
- By the main result, the optimal CDS strategy is given by

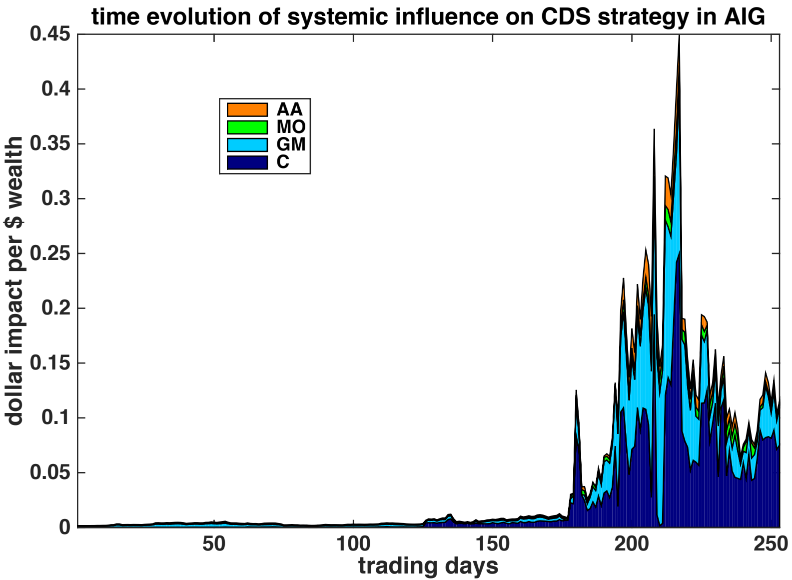
$$\psi(t) = \boldsymbol{\Theta}^{-1}\Big(\big(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\big)^{-1} + \boldsymbol{\Lambda}^{-1}\Big)(h_n - \lambda_n)_{n\in\mathbb{M}(t)} + \boldsymbol{\Theta}^{-1}\big(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\big)^{-1}(\mu_n - r)_{n\in\mathbb{M}(t)},$$

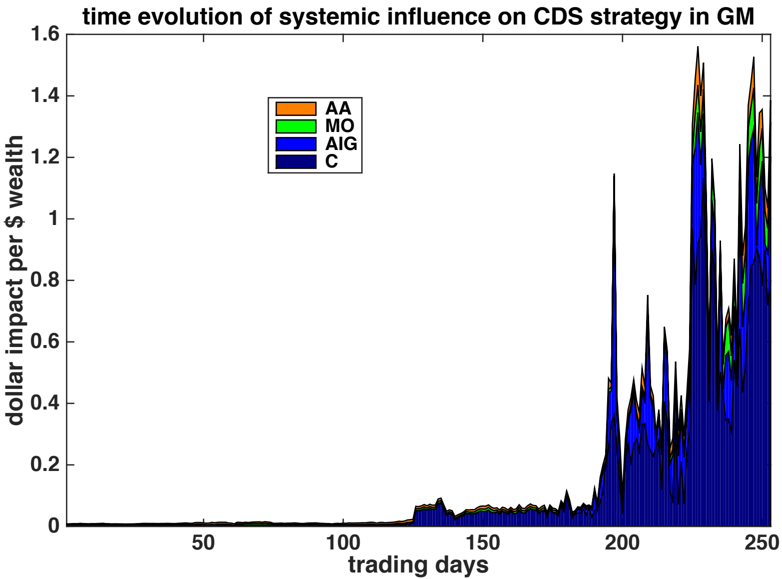
  where $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_n)_{n\in\mathbb{M}(t)}$

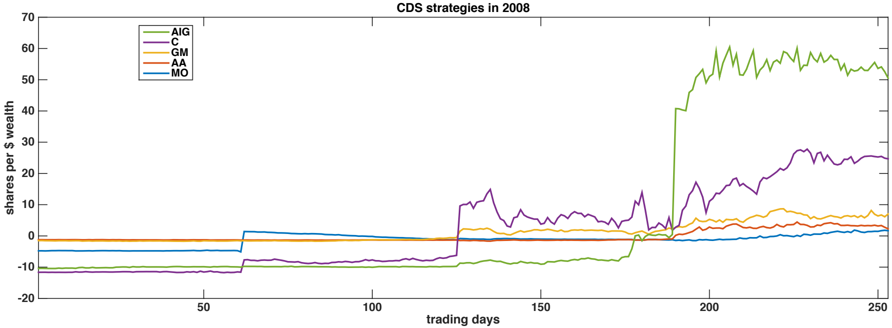
  $\Longrightarrow$ a change in the default premium $h_n - \lambda_n$ of company n also affects the CDS strategy $\psi_i(t)$ in company i.
- We refer to the total effect of $h_n - \lambda_n$ of companies $n \neq i$ onto $\psi_i(t)$ as systemic influence on the CDS strategy in company i.



- For each company i, the bar can be split into 29 terms contributing to the aggregate systemic influence $\sum_{n \neq i} \psi^{(n)}$ on the CDS strategy referencing the i^{th} company.
- The biggest contributions come from the companies C (Citigroup), AIG, GM (General Motors Company), MO (Altria Group), and AA (Alcoa).


time evolution of systemic influence on CDS strategy in AIG
AA
MO
GM
C
dollar impact per $ wealth
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
50
100
150
200
250
trading days

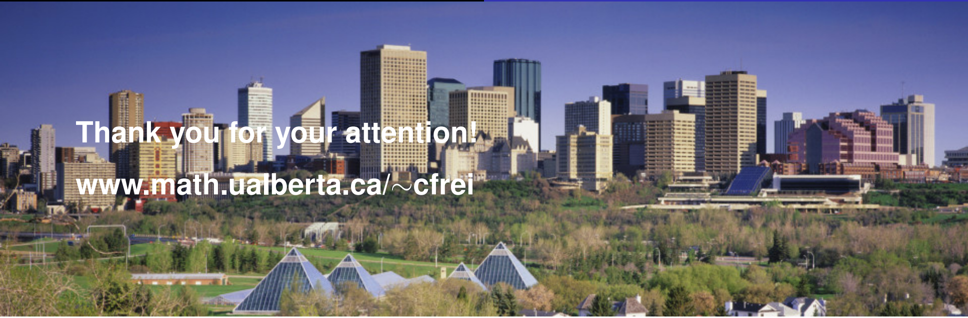

time evolution of systemic influence on CDS strategy in GM
AA
MO
AIG
C
dollar impact per $ wealth
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
50
100
150
200
250
trading days



- The investor is most active on the CDS market during the distressed last quarter of 2008.
- The investor shifts from selling small amounts of protection on the five systemically most influential companies (in the first quarter) to purchasing high amounts of protection on these companies in the last quarter of 2008.

Summary

1. Developed an equity-credit portfolio framework with interacting market and default risk.
2. Provided closed-form expressions for the optimal investment strategies in stocks and CDSs despite interrelations of risk factors.
3. Introduced a calibration procedure for our model which progressively refines the systemic component.
4. Analyzed systemic influences in investment decisions in the calibrated model for the DJIA stocks in 2008.
5. Revealed a small number of (mostly financial) companies with high systemic influences as well as companies which are highly sensitive to systemic influences.



Thank you for your attention!

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Announcements: Events this summer at the University of Alberta in Edmonton, Canada

1. Summer School in Mathematical Finance, June 25–July 6
Lectures on imperfect financial markets and algorithmic trading

2. Sixth International IMS-FIPS Workshop, July 7–9
Conference on stochastic applications to finance and insurance

For more information, please see **www.mathfinance2016.com**

References

- Becherer and Schweizer: Classic solutions to reaction-diffusion systems for hedging problems with interacting Itô and point processes. *Annals of Applied Probability*, 2008.
- Bo and Capponi: Optimal investment in credit derivatives portfolio under contagion risk. *Mathematical Finance*, forthcoming.
- Capponi and Frei: Systemic influences on optimal equity-credit investment. *Management Science*, forthcoming.
- Duffie, Saita and Wang: Multi-period corporate default prediction with stochastic covariates. *Journal of Financial Economics*, 2007.
- Giesecke, Kim, Kim and Tsoukalas: Optimal credit swap portfolios. *Management Science*, 2014.
- Gordon, Paradis and Rorke: Experimental evidence on alternative portfolio decision rules. *American Economic Review*, 1972.