Stochastic representation for solutions to nonlocal Bellman equations

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Outline

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   - Stochastic control problem
2. Stochastic Representation Formula
   - Known results
   - Assumptions
   - Representation formula for smooth value function
   - Representation formula in finite control set
   - Representation formula in general control set

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Stochastic control problem

- We fix a terminal time $T > 0$, a Polish space $\mathcal{U}$ (the control space) and a bounded domain O in $\mathbb{R}^d$.
- For any $0 \leq t < T$, let μ be a generalized reference probability space $(\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \mathcal{L})$ where $\mathcal{W}$ is an m_1-dimensional standard Brownian motion and $\mathcal{L}$ is an m_2 dimensional Lévy process.
- Let $\mathcal{A}_\mu$ be the set of all $\mathcal{F}_s^t$-predictable $\mathcal{U}$-valued processes on $[t, T]$, and let $\mathcal{A}_t := \cup_\mu \mathcal{A}_\mu$.

Stochastic control problem

For any $x \in O$ and $U(\cdot) \in \mathcal{A}_\mu$, we consider

$$\begin{aligned} X(s) &= x + \int_t^s b(r, X(r), U(r))dr + \int_t^s \sigma(r, X(r), U(r))d\mathcal{W}(r) \\ &\quad + \int_t^s \int_{\mathbb{R}^{m_2}} \gamma(r, X(r-), U(r), z)\tilde{N}(dr, dz), \end{aligned}$$

where $\tilde{N}$ is a compensated Poisson random measure. Let $\tau = T \wedge \inf\{s \geq t; X(s) \notin O\}$ and

$$J_\mu(t, x; U) := \mathbb{E}\left(\int_t^\tau \Gamma(s, X(s), U(s))ds + \Psi(\tau, X_\tau)\right). \qquad (1)$$

Stochastic control problem

We will consider the stochastic control problem by first taking the infinmum of the cost functional (1) over all $U \in \mathcal{A}_\mu$, i.e.

$$V_\mu(t,x) := \inf_{U \in \mathcal{A}_\mu} J_\mu(t,x;U), \tag{2}$$

and then by taking the infinmum of (2) over all generalized reference probability spaces, i.e.

$$V(t,x) := \inf_\mu V_\mu(t,x).$$

Stochastic control problem

The corresponding nonlocal HJB equation is then given by

$$\inf_{u \in \mathcal{U}} \left(\mathcal{A}^u W(t,x) + \Gamma(t,x,u) \right) = 0 \quad \text{in } Q := [0,T) \times O, \tag{3}$$

with terminal-boundary condition

$$W(t,x) = \Psi(t,x) \quad \text{on } \partial_{np} Q := ([0,T) \times O^c) \cup \left(\{T\} \times \mathbb{R}^d \right), \tag{4}$$

where

$$\begin{aligned} &\mathcal{A}^u W(t,x) \\ &:= \partial_t W(t,x) + \mathrm{tr}\left(a(t,x,u) D_x^2 W(t,x) \right) + b(t,x,u) \cdot D_x W(t,x) \\ &+ \int_{\mathbb{R}^{m_2}} W(t,x+\gamma(t,x,u,z)) - W(t,x) - D_x W(t,x) \cdot \gamma(t,x,u,z) \nu(dz) \end{aligned}$$

and where $a(t,x,u) := \frac{1}{2}\sigma(t,x,u)\sigma^T(t,x,u)$.

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Known results

- Stochastic representation formula for PDEs: Buckdahn, Fleming, Krylov, Ma, Li, Lions, Soner, Souganidis, Swiech, Touzi, Zhang...
- Stochastic representation formula for integro-PDEs: Barles, Biswas, Buckdahn, Caffarelli, Li, Kharroubi, Pham, Soner, Swiech...

Outline

Assumptions on the coefficients

- γ is measurable, Ψ is bounded and continuous, and b, σ, Γ are bounded and uniformly continuous.
- There exist a universal constant $C > 0$, a modulus of continuity ϖ and a measurable function ρ satisfying $\int_{\mathbb{R}^{m_2}} \rho^2(z)\nu(dz) < +\infty$ such that for any $u, u_1, u_2 \in \mathcal{U}$, $s, s_1, s_2 \in [0, T)$, $y_1, y_2 \in \mathbb{R}^d$ and $z \in \mathbb{R}^{m_2}$

$$|||\gamma(\cdot,\cdot,\cdot,z)|||_{L^\infty([0,T]\times\mathbb{R}^d\times\mathcal{U})} \le C\rho(z)$$

$$|b(s,y_1,u)-b(s,y_2,u)|+\|\sigma(s,y_1,u)-\sigma(s,y_2,u)\| \le C|y_1-y_2|,$$

$$\begin{aligned} & |\gamma(s_1,y_1,u_1,z)-\gamma(s_2,y_2,u_2,z)| \\ \le \quad & C\rho(z)\left(\varpi(|u_1-u_2|+|s_1-s_2|)+|y_1-y_2|\right) \end{aligned}$$

Assumptions on the domain

- O is η-prox-regular for some fixed $\eta > 0$, i.e. for any $x \in \partial O$ and any unit vector $n \in N(O, x)$ we have

$$B_\eta(x + \eta n) \cap O = \emptyset$$

where the proximal normal cone to O at $x \in \partial O$ by

$$N(O, x) := \{n \in \mathbb{R}^d; \text{there exists } l > 0 \text{ such that } x \in P(O, x + ln)\}$$

and where

$$P(O, y) = \left\{ z \in \partial O; \inf_{p \in O} |p - y| = |z - y| \right\}.$$

Parabolicity Assumptions

- There exists a constant $\lambda > 0$ such that for any $x \in \partial O$ and $n_x \in N(O, x)$ we have

$$n_x \sigma(t, x, u) \sigma^T(t, x, u) n_x^T \geq \lambda, \quad \text{for any } t \in [0, T) \text{ and } u \in \mathcal{U}.$$

Outline

Representation formula for smooth value function

Theorem (Gong, Mou and Swiech, 2017)

Let $W \in C^{1,2}([0,T] \times \mathbb{R}^d)$ be a solution to nonlocal HJB with terminal-boundary condition $u = \Psi$ on $\partial_{np} Q$. Then we have

$$W(t,x) = V(t,x) = V_\mu(t,x)$$

for any generalized reference probability space μ and any stopping time θ, with $\theta \in [t,T]$ $\mathbb{P}$-a.s.

$$W(t,x) = \inf_{U \in \mathcal{A}_\mu} \mathbb{E} \left(\int_t^{\theta \wedge \tau} \Gamma(s, X(s), U(s)) ds + W(\theta \wedge \tau, X(\theta \wedge \tau)) \right).$$

Outline

Approximation

Let $\mathcal{U}$ be a finite control set, $O_\delta := \{y \in \mathbb{R}^d; \mathrm{dist}(y, O) < \delta\}$ and $Q_\delta := [0, T+\delta] \times O_\delta$.
We construct sequences of functions b_n, σ_n, γ_n and Γ_n, $n \in \mathbb{N}$ satisfying

- b_n, σ_n and Γ_n are uniformly bounded.
- There exist $C > 0$ and $C_n > 0$, depending on n such that for any $u \in \mathcal{U}$, $s_1, s_2 \in [0, T+1)$, $y_1, y_2 \in \mathbb{R}^d$ and $z \in \mathbb{R}^{m_2}$

$$\begin{aligned} & |b_n(s_1, y_1, u) - b_n(s_2, y_2, u)| + \|\sigma_n(s_1, y_1, u) - \sigma_n(s_2, y_2, u)\| \\ + \quad & |\Gamma_n(s_1, y_1, u) - \Gamma_n(s_2, y_2, u)| \le C_n(|s_1 - s_2| + |y_1 - y_2|) \end{aligned}$$

$$|\gamma_n(s_1, y_1, u, z) - \gamma_n(s_2, y_2, u, z)| \le C_n \rho(z)(|s_1 - s_2| + |y_1 - y_2|)$$

$$\|\gamma_n(\cdot, \cdot, \cdot, z)\|_{L^\infty([0,T+1]\times\mathbb{R}^d\times U)} \le C\rho(z).$$

Approximation

- As $n \to +\infty$,

$$\|b - b_n\|_{L^\infty([0,T]\times\mathbb{R}^d\times U)} \to 0,$$

$$\|\sigma - \sigma_n\|_{L^\infty([0,T]\times\mathbb{R}^d\times U)} \to 0,$$

$$\|\Gamma - \Gamma_n\|_{L^\infty([0,T]\times\mathbb{R}^d\times U)} \to 0,$$

and

$$\int_{\mathbb{R}^{m_2}} \|\gamma(\cdot,\cdot,\cdot,z) - \gamma_n(\cdot,\cdot,\cdot,z)\|^2_{L^\infty([0,T]\times\mathbb{R}^d\times U)}\nu(dz) \to 0.$$

- For $\delta > 0$ and $x \in \partial O_\delta$, there is a unit vector $n_{x,\delta} \in N(O_\delta, x)$ such that $\overline{B}_{\eta/2}(x + \frac{\eta}{2} n_{x,\delta}) \cap \overline{O}_\delta = \{x\}$.Moreover, for any $t \in [0, T+\delta)$, $u \in \mathcal{U}$ and $n \in \mathbb{N}$ large enough

$$n_{x,\delta}\sigma_n(t,x,u)\sigma_n^T(t,x,u)n_{x,\delta}^T \geq \frac{\lambda}{2}.$$

Approximation

- Let $\{\epsilon_n\}_n$ be a sequence of positive real numbers such that $\lim_n \epsilon_n = 0$. For any $\mu_1 := (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \tilde{\mathcal{W}}, \mathcal{L})$, $U \in \mathcal{A}_{\mu_1}$ and $x \in \mathbb{R}^d$, we consider the following SDE:

$$\begin{aligned} X_n(s) &= x + \int_t^s b_n(r, X_n(r), U(r))dr + \int_t^s \sigma_n(r, X_n(r), U(r))d\mathcal{W}(r) \\ &\quad + \int_t^s \sqrt{\epsilon_n} d\tilde{\mathcal{W}}(r) + \int_t^s \int_{\mathbb{R}^{m_2}} \gamma_n(r, X_n(r-), U(r), z)\tilde{N}(dr, dz). \end{aligned}$$

- Let $\tau_{\delta,n} := T \wedge \inf\{s \geq t; X_n(s) \notin O_{\frac{\delta}{2}}\}$.

Existence of a unique smooth solution

Theorem (Mou and Swiech, 2017)

Let $\mathcal{U}$ be a finite set and $\Psi \in C_b^{1+\alpha/2,\,2+\alpha}([0,\,T+1]\times\mathbb{R}^d)$ for some $\alpha > 0$. Then, there exists a unique viscosity solution

$$W_{\delta,n} \in C_{\mathrm{loc}}^{1+\alpha/2,2+\alpha}(Q_\delta) \cap \mathrm{Lip}_b\left([0,\,T+\delta]\times\mathbb{R}^d\right)$$

to

$$\begin{cases} \inf_{u\in\mathcal{U}}\left(\mathcal{A}^u_{\delta,n}W_{\delta,n}(t,x)+\Gamma_n(t,x,u)\right)=0 \quad \text{in } Q_\delta, \\ W_{\delta,n}(t,x)=\Psi(t,x), \quad (t,x)\in\partial_{\mathrm{np}}Q_\delta, \end{cases}$$

where

$$\begin{aligned} &\mathcal{A}^u_{\delta,n}W_{\delta,n} \\ :=\ &\frac{\partial W_{\delta,n}}{\partial t}+b_n\cdot D_xW_{\delta,n}(t,x)+\frac{1}{2}\mathrm{tr}\left((a_n+\epsilon_n I_n)\,D_x^2W_{\delta,n}\right) \\ &+\int_{\mathbb{R}_0^{m_2}}\left(W_{\delta,n}\left(t,x+\gamma_n\right)-W_{\delta,n}(t,x)-D_xW_{\delta,n}(t,x)\cdot\gamma_n\right)\nu(dz). \end{aligned}$$

Approximation

It follows from the representation formula for the smooth value function with $\theta_U = \tau \wedge \tau_{\delta,n}$, that

$$W_{\delta,n}(t,x) = \inf_{U \in \mathcal{A}_{\mu_1}} \mathbb{E}\left(\int_t^{\tau \wedge \tau_{\delta,n}} \Gamma_n\left(s, X_n(s), U(s)\right) ds \right.$$
$$\left. + W_{\delta,n}\left(\tau \wedge \tau_{\delta,n}, X_n(\tau \wedge \tau_{\delta,n})\right) \right).$$

Existence a viscosity solution

Theorem (Mou, Anal. PDE 2017)

There exists a unique viscosity solution

$$W_\delta \in C_b([0, T+\delta] \times \mathbb{R}^d)$$

to

$$\begin{cases} \inf_{u \in \mathcal{U}} \left(\mathcal{A}^u W_\delta(t,x) + \Gamma(t,x,u) \right) = 0 \quad \textit{in} \;\; Q_\delta, \\ W_{\delta,n}(t,x) = \Psi(t,x), \quad (t,x) \in \partial_{\mathrm{np}} Q_\delta. \end{cases}$$

Representation formula in finite control set

Theorem (Gong, Mou and Swiech, 2017)

Let $\mathcal{U}$ be a finite set and $\Psi \in C_b^{1+\alpha/2,\, 2+\alpha}([0, T+1] \times \mathbb{R}^d)$ for some $\alpha > 0$. For each $t \in [0, T]$, let $\mu_1 = (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \widetilde{\mathcal{W}}, \mathcal{L})$ and set $\mu = (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \mathcal{L})$. Then, for any $x \in \overline{O}$,

$$W_\delta(t, x) = \inf_{U \in \mathcal{A}_\mu} \mathbb{E}\left(\int_t^\tau \Gamma(s, X(s), U(s))\, ds + W_\delta(\tau, X(\tau)) \right).$$

Outline

Approximation

- Let $\mathcal{U}$ be a Polish space, and $\mathcal{U}_n := \{v_1, ..., v_n\}$ where $\{v_i\}_{i\in\mathbb{N}}$ is a countable dense subset of $\mathcal{U}$.
- For any $\mu_1 = (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \tilde{\mathcal{W}}, \mathcal{L})$, we let $\mu = (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \mathcal{L})$ and $\mathcal{A}_\mu^n$ be the collection of all predictable $\mathcal{U}_n$-valued process on $[t, T]$.
- For any $U_n \in \mathcal{A}_\mu^n$ and $x \in \mathbb{R}^d$, we consider the following SDE:

$$\begin{aligned}\bar{X}_n(s) &= x + \int_t^s b(r, \bar{X}_n(r), U_n(r))dr + \int_t^s \sigma(r, \bar{X}_n(r), U_n(r))d\mathcal{W}(r) \\ &\quad + \int_t^s \int_{\mathbb{R}^{m_2}} \gamma(r, \bar{X}_n(r-), U_n(r), z)\tilde{N}(dr, dz).\end{aligned}$$

- Let $\bar{\tau}_{\delta,n} := T \wedge \inf\{s \geq t; X(s) \notin O_{\frac{\delta}{2}}\}$.

A crucial estimate

Lemma (Gong, Mou and Swiech)

For any $U \in \mathcal{A}_\mu$, *there exists* $\{U_{n_k}\}_{k\in\mathbb{N}}$, *where* $U_{n_k} \in \mathcal{A}_\mu^{n_k}$, *such that for any* $x \in \mathbb{R}^d$ *and as* $k \to +\infty$,

$$\mathbb{E}\left(\int_t^T \left|\Gamma\left(s, X(s), U_{n_k}(s)\right) - \Gamma\left(s, X(s), U(s)\right)\right|^2 ds\right) \to 0,$$

and

$$\mathbb{E}\left(\sup_{\ell\in[t,T]} \left|X(\ell) - \overline{X}_{n_k}(\ell)\right|^2\right) \to 0.$$

Representation formula in general control set

Theorem (Gong, Mou and Swiech, 2017)

Let W be the viscosity solution to nonlocal HJB (3) with terminal-boundary condition $W = \Psi$ on $\partial_{np}Q$. Let $t \in [0, T]$, let $\mu_1 = (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \widetilde{\mathcal{W}}, \mathcal{L})$ and set $\mu = (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, \mathcal{W}, \mathcal{L})$. Then for any $x \in \overline{O}$

$$W(t,x) = \inf_{U \in \mathcal{A}_\mu} \mathbb{E}\left(\int_t^\tau \Gamma\left(s, X(s), U(s)\right) ds + W(\tau, X(\tau)) \right).$$

Dynamic programming principle

Theorem (Gong, Mou and Swiech, 2017)

With the assumptions above, we have for any $x \in \overline{O}$ and stopping time θ, with $\theta \in [t, T]$ $\mathbb{P}$-a.s.

$$W(t,x) = \inf_{U \in \mathcal{A}_\mu} \mathbb{E}\left(\int_t^{\theta \wedge \tau} \Gamma\left(s, X(s), U(s)\right) ds + W(\theta \wedge \tau, X(\theta \wedge \tau)) \right).$$

Dynamic programming principle over all reference spaces

Corollary (Gong, Mou and Swiech, 2017)

With the assumptions above, we have for any $x \in \overline{O}$ *and stopping time* θ*, with* $\theta \in [t, T]$ $\mathbb{P}$*-a.s.*

$$W(t,x) = \inf_{U \in \mathcal{A}_t} \mathbb{E}\left(\int_t^{\theta \wedge \tau} \Gamma\left(s, X(s), U(s)\right) ds + W(\theta \wedge \tau, X(\theta \wedge \tau)) \right).$$

Thank you for your attention!