Large Deviations from the Hydrodynamic Limit for a System with Nearest Neighbor Interactions

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Background

- Let $\{Y_i\}_{i\geq 1}$ be a sequence of $\mathbb{R}^d$-valued iid zero mean random variables with common probability law ρ.
- Let $S_n = \sum_{i=1}^n Y_i$. Then $S_n/n \to 0$ a.s. by LLN.
- Large Deviation Principle: For $c > 0$

$$\mathbb{P}(|S_n| > nc) \approx \exp\{-n \inf\{I(y) : |y| \geq c\}\},$$

where for $y \in \mathbb{R}^d$,

$$I(y) = \sup_{\alpha \in \mathbb{R}^d} \{\langle \alpha, y\rangle - \log \int_{\mathbb{R}^d} \exp\langle \alpha, y\rangle \rho(dy)\}.$$

Large Deviation Principle.

Definition. Consider a sequence $\{X^{\varepsilon}\}_{\varepsilon>0}$ of $\mathcal{E}$ valued r.vs.

- $I : \mathcal{E} \to [0,\infty]$ is a rate function on $\mathcal{E}$ if for each $M < \infty$, $\{x \in \mathcal{E} : I(x) \leq M\}$ is compact.
- $\{X^{\epsilon}\}$ is said to satisfy the large deviation principle on $\mathcal{E}$ (as $\varepsilon \to 0$) with rate function I if:
  - For each closed $F \subset \mathcal{E}$
  $$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^{\epsilon} \in F) \leq -\inf_{x\in F} I(x).$$
  - For each open $G \in \mathcal{E}$
  $$\liminf_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^{\epsilon} \in G) \geq -\inf_{x\in G} I(x).$$

Formally, for small ε:

$$\mathbb{P}(X^{\epsilon} \in A) \approx \exp\left\{-\frac{\inf_{x\in A} I(x)}{\varepsilon}\right\}, \ A \in \mathcal{B}(\mathcal{E}).$$

Stochastic Control Connection (Fleming 1978)

Consider a small noise n-dimensional SDE:

$$dX^\varepsilon(t) = b(X^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(X^\varepsilon(t))dW(t),\ X^\varepsilon(0) = x.$$

- b, σ suitable coefficients... W a f.d. BM.
- Let $G \subset \mathbb{R}^n$ be bounded open. Let $x \in G$ and $\tau^\varepsilon = \inf\{t : X^\varepsilon(t) \in \partial G\}$.
- Interested in $\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(X^\varepsilon(\tau^\varepsilon) \in N)$, where $N \subset \partial G$.

Stochastic Control Connection (Ctd.)

- Formally, with Φ a nonnegative C^2 function, $\Phi(x) \approx M 1_{N^c}(x)$, M a large scaler,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(X^\varepsilon(\tau^\varepsilon) \in N) \approx \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x \left\{ e^{-\Phi(X^\varepsilon(\tau^\varepsilon))/\varepsilon} \right\}.$$

- Then $g^\varepsilon(x) = \mathbb{E}_x \left\{ e^{-\Phi(X^\varepsilon(\tau^\varepsilon))/\varepsilon} \right\}$ solves

$$\begin{cases} \mathcal{L}^\varepsilon g^\varepsilon(x) = 0, & x \in G \\ g^\varepsilon(x) = e^{-\Phi(x)/\varepsilon}, & x \in \partial G \end{cases}$$

  where $\mathcal{L}^\varepsilon g = \frac{\varepsilon}{2} \mathrm{Tr}(\sigma D^2 g \sigma') + b \cdot \nabla g$.

- Interested in asymptotics of $-\varepsilon \log g^\varepsilon$.

Stochastic Control Connection (ctd.)

- **log transform**: Let $J^{\varepsilon} = -\varepsilon \log g^{\varepsilon}$. Then J^{ε} solves

$$\frac{\varepsilon}{2}\mathrm{Tr}(\sigma D^2 J^{\varepsilon}\sigma') + H(x, \nabla J^{\varepsilon}) = 0$$

  where

$$H(x,p) = \min_{v\in\mathbb{R}^n}[L(x,v) + p\cdot v],\ x \in G,\ p \in \mathbb{R}^n$$

  and $L(x,v) = \frac{1}{2}(b(x) - v)'[\sigma(x)\sigma'(x)]^{-1}(b(x) - v)$.

- J^{ε} can be characterized as the value function of the stochastic control problem:

$$\begin{aligned} J^{\varepsilon}(x) &= \inf_{u\in\mathcal{A}} \mathbb{E}_x \left\{ \int_0^{\tilde{\tau}^{\varepsilon}} L(\tilde{X}^{\varepsilon}(t), u(t))dt + \Phi(\tilde{X}^{\varepsilon}(\tilde{\tau}^{\varepsilon})) \right\} \\ d\tilde{X}^{\varepsilon}(t) &= u(t)dt + \sqrt{\varepsilon}\sigma(X^{\varepsilon}(t))dW(t),\ \tilde{X}^{\varepsilon}(0) = x \end{aligned}$$

Stochastic Control Connection (ctd.)

- One can argue $J^{\varepsilon} \to J$, where $J(x)$ is the value function of the deterministic control problem:

$$J(x) = \inf_{\phi,\theta} \left[\int_0^{\theta} L(\phi(t), \dot{\phi}(t))dt + \Phi(\phi(\theta)) \right],$$

  where inf is over all abs. cts. ϕ such that $\phi(0) = x$, and $\theta = \inf\{t : \phi(t) \in \partial G\}$.

- Later works: Sheu (1985), Dupuis and Ellis(1997), Feng and Kurtz (2005).

LDP and Laplace Principle.

- LDP is equivalent to Laplace principle if the state space is Polish (Varadhan(1966), Bryc(1990)):
  - A collection of $\mathcal{E}$ valued random variables $\{X^\varepsilon\}$ is said to satisfy Laplace principle with rate function I, if for all $h \in C_b(\mathcal{E})$

$$\lim_{\epsilon \to 0} -\epsilon \log \ \mathbb{E}\left\{\exp\left[-\frac{1}{\epsilon}h(X^\epsilon)\right]\right\} = \inf_{x \in \mathcal{E}}\{h(x) + I(x)\}.$$

- From Donsker-Varadhan:

$$-\epsilon \log \ \mathbb{E}\Big\{\exp\Big[-\tfrac{1}{\epsilon}h(X^\epsilon)\Big]\Big\} = \inf_{Q \in \mathcal{P}(\mathcal{E})}\Big[\int h(x)dQ(x) + R(Q\|P^\varepsilon)\Big].$$

LDP and Laplace Principle.

- Goal is to show the convergence of variational expressions:

$$\inf_{Q\in\mathcal{P}(E)}\left[\int h(x)dQ(x)+R(Q\|P^{\varepsilon})\right] \xrightarrow{\varepsilon\to 0} \inf_{x\in\mathcal{E}}\{h(x)+I(x)\}.$$

Some Settings where the Approach Works.

- Small Noise SPDE (B., Dupuis and Maroulas (2008)).
- Stochastic Flows of Diffeomorphisms (B., Dupuis and Maroulas (2010)).
- Finite and Infinite Dimensional Jump-Stochastic Dynamical Systems with Small Noise (B., Chen and Dupuis (2013)).
- Moderate deviation principles for SDE w/ Jumps in Finite and Infinite Dimensions (B., Dupuis and Ganguly (2016)).
- Component Size Large Deviations for Configuration Model (Bhamidi, B., Dupuis and Wu (2017)).
- Multiscale jump-diffusions – Large Deviations from Stochastic Averaging Principle (B., Dupuis and Ganguly (2017)).
- Weakly Interacting Diffusions – Large and Moderate Deviations (B., Dupuis and Fischer (2012), B. and Wu (2016)).

A System with Nearest Neighbor Interactions.

- Ginzburg-Landau in Finite Volume: For $t \in [0, T]$ and $i = 1, \dots N$

$$dX_i^N(t) = \frac{N^2}{2}\left[\phi'\left(X_{i-1}^N(t)\right) - 2\phi'\left(X_i^N(t) + \phi'\left(X_{i+1}^N(t)\right)\right)\right] dt$$
$$+ N\left[dB_i(t) - dB_{i+1}(t)\right]$$

- $\{1/N, \dots, (N-1)/N, 1\}$ is the periodic lattice. I.e. identify X_{N+1}^N with X_1^N.

- $\{B_i(t)\}_{i=1}^{\infty}$ are independent standard one-dimensional Brownian motions given on some probability space $(\mathcal{V}, \mathcal{F}, \mathbb{P})$.

A System with Nearest Neighbor Interactions.

- $\phi : \mathbb{R} \to \mathbb{R}$ is C^2 and

$$\int_{\mathbb{R}} \exp(-\phi(x))dx = 1, \; M(\lambda) \doteq \int_{\mathbb{R}} \exp(\lambda x - \phi(x)) < \infty$$

for all $\lambda \in \mathbb{R}$, and for all $\sigma < \infty$

$$\int_{\mathbb{R}} \exp(\sigma|\phi'(x)| - \phi(x))dx < \infty.$$

Model Description (ctd.)

- Invariant measure for X^N: $\Phi^N(dx) \doteq \Phi(dx_1)\Phi(dx_2)\ldots\Phi(dx_N)$ where $\Phi(dx) \doteq e^{-\phi(x)}dx$.

- Consider X^N with $X^N(0) \sim \Phi^N$ and process μ^N with values in $\mathcal{M}_S$ (the space of signed measures on the unit circle S):

$$\mu^N(t, d\theta) \doteq \frac{1}{N}\sum_{i=1}^{N} X_i^N(t)\delta_{i/N}(d\theta).$$

- A LLN for μ^N shown in Guo-Papanicolaou-Varadhan (1988) and a LDP proved in Donsker-Varadhan (1989).

- A new proof...

Remarks

- The original proof [DW(1989)] requires control on exponential moments and exponential probability estimates. This approach has been extended to many different systems.
- Exponential estimates are the hardest parts of the proof.
- The new proof uses stochastic control representations and weak convergence methods.
- Proof techniques similar to that for LLN analysis. No exponential estimates are invoked.
- Key Technical Step: Suitable Regularity of Densities of Controlled Processes. Bounds on certain Dirichlet Forms.

Main Result

- Let $\mathcal{M}_S^l$ be elements in $\mathcal{M}_S$ with total variation bounded by l.
- Let for $l \in \mathbb{N}$, $\Omega_l \doteq C([0,T] : \mathcal{M}_S^l)$ the Polish space of continuous paths of signed measures with total variation bounded by l.
- Then $\Omega \doteq C([0,T] : \mathcal{M}_S) = \cup_{l \in \mathbb{N}} C([0,T] : \mathcal{M}_S^l) = \cup_{l \in \mathbb{N}} \Omega_l$. This space is equipped with the direct limit topology.
- Theorem $\{\mu^N\}$ satisfies a LDP in $C([0,T] : \mathcal{M}_S)$ with rate function I.

Rate Function

- Let

$$\rho(\lambda) \doteq \log M(\lambda),\ \lambda \in \mathbb{R},\ h(x) \doteq \sup_{\lambda \in \mathbb{R}} \{\lambda x - \rho(\lambda)\}.$$

- Let $\tilde{\Omega}$ be the collection of all μ in Ω such that for all t, $\mu(t, d\theta) = m(t, \theta) d\theta$ and m satisfies

$$\int_{[0,T] \times S} [h(m(t,\theta)) + [h'(m(t,\theta))]_\theta^2] dt d\theta < \infty,$$

- Let $\mathcal{P}_*(\mathbb{R} \times S)$ be all $\pi \in \mathcal{P}(\mathbb{R} \times S)$ such that

$$\pi(dx\ d\theta) = \pi_1(dx \mid \theta) d\theta,$$

with

$$m_0(\theta) = \int_{\mathbb{R}} x \pi_1(dx|\theta),\ \int_S h(m_0(\theta)) d\theta < \infty.$$

Rate Function

- For $u \in L^2([0,T] \times S : \mathbb{R})$ and $\pi \in \mathcal{P}_*(\mathbb{R} \times S)$ let $\mathcal{M}_\infty(u,\pi)$ be all $\mu \in \tilde{\Omega}$, s.t. $\mu(t, d\theta) = m(t,\theta)d\theta$, and m solves weakly

$$\partial_t m(t,\theta) = \frac{1}{2}\left[h'(m(t,\theta))\right]_{\theta\theta} - \partial_\theta u(t,\theta),\ m(0,\theta) = m_0(\theta)$$

- Letting $\pi_0(dx\, d\theta) = \Phi(dx)d\theta$, define $I : \Omega \to [0,\infty]$ by

$$I(\mu) = \inf_{\{(u,\pi):\mu \in \mathcal{M}_\infty(u,\pi)\}} \left[\frac{1}{2}\int_0^T \int_S |u(s,\theta)|^2 d\theta ds + R(\pi \| \pi_0)\right]$$

for $\mu \in \tilde{\Omega}$, and set $I(\mu) = \infty$ otherwise.

Main Steps in Proof

- Compact level sets: I is a rate function on Ω, namely for every $M < \infty$ $\{\mu \in \Omega : I(\mu) \leq M\}$ is compact.

- Laplace upper bound: For all $F \in C_b(\Omega)$

$$\limsup_{N\to\infty} -\frac{1}{N} \log \mathbb{E}[\exp(-NF(\mu^N))] \geq \inf_{\mu\in\Omega}\{F(\mu) + I(\mu)\}.$$

- Laplace lower bound: For all $F \in C_b(\Omega)$

$$\liminf_{N\to\infty} -\frac{1}{N} \log \mathbb{E}[\exp(-NF(\mu^N))] \leq \inf_{\mu\in\Omega}\{F(\mu) + I(\mu)\}.$$

Variational Representation

- Boué-Dupuis (1998), B., Fan and Wu (2017).
- Let $(\bar{\mathcal{V}}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be a probability space with an N-dimensional Brownian motion, $\mathbf{B}^N = (B_1, \ldots B_N)$, and a $\mathbb{R}^N$-valued random variable $\bar{X}^N(0)$ independent of $\mathbf{B}^N$ and with probability law Π^N.
- Let $\{\bar{\mathcal{F}}_t\}$ be any filtration satisfying the usual conditions such that $\mathbf{B}^N$ is a $\{\bar{\mathcal{F}}_t\}$-Brownian motion and $\bar{X}^N(0)$ is $\bar{\mathcal{F}}_0$ measurable.
- Let $\mathcal{K}_{\Pi^N} \doteq (\bar{\mathcal{V}}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}, \bar{\mathbb{P}}, \bar{X}^N(0), \mathbf{B}^N)$ and let

$$\mathcal{A}^N(\mathcal{K}_{\Pi^N}) \doteq \{\psi : \psi = (\psi_i)_{i=1}^N, \psi_i \text{ is simple and } \bar{\mathcal{F}}_t \text{ adapted}\}.$$

Variational Representation

- For a $\psi^N \in \mathcal{A}^N(\mathcal{K}_{\Pi^N})$, let

$$\bar{B}_i^N(t) \doteq B_i(t) + \int_0^t \psi_i^N(s)ds,\ \ t \in [0, T],\ \ i = 1, \ldots N.$$

Let

$$d\bar{X}_i^N(t) = \frac{N^2}{2}\left[\phi'\left(\bar{X}_{i-1}^N(t)\right) - 2\phi'\left(\bar{X}_i^N(t) + \phi'\left(\bar{X}_{i+1}^N(t)\right)\right)\right]dt \\ + N\left[d\bar{B}_i(t) - d\bar{B}_{i+1}(t)\right]$$

- Disintegrate $\Pi^N \in \mathcal{P}(\mathbb{R}^N)$, as

$$\Pi^N(dx) \doteq \Pi_1(dx_1)\Pi_2(dx_2|x_1)\ldots\Pi_N(dx_N|dx_1,\ldots,dx_{N-1}) \doteq \prod_{i=1}^N \bar{\Phi}_i^N(x, dx_i),$$

and with $\bar{X}^N(0)$ distributed as Π^N, let $\bar{\mathbf{\Phi}}_i^N(dz) \doteq \bar{\Phi}_i^N(\bar{X}^N(0), dz)$.

Variational Representation

Let $F \in C_b(\Omega)$. Then for all $N \in \mathbb{N}$

$$-\frac{1}{N}\log \mathbb{E}\exp(-NF(\mu^N))$$

$$= \inf_{\Pi^N, \mathcal{K}_{\Pi^N}} \inf_{\psi^N \in \mathcal{A}^N(\mathcal{K}_{\Pi^N})} \bar{\mathbb{E}}_{\Pi^N}\left[\frac{1}{N}\sum_{i=1}^{N}\left(R(\bar{\mathbf{\Phi}}_i^N \| \Phi) + \frac{1}{2}\int_0^T |\psi_i^N(s)|^2 ds\right) + F(\bar{\mu}^N)\right].$$

Laplace Upper Bound

- Fix $F \in C_b(\Omega)$ and let $\epsilon \in (0,1)$. Choose for each $N \in \mathbb{N}$, $\Pi^N \in \mathcal{P}(\mathbb{R}^N)$, a system $\mathcal{K}_{\Pi^N}$ and $\psi^N \in \mathcal{A}^N(\mathcal{K}_{\Pi^N})$ such that

$$-\frac{1}{N}\log \mathbb{E}\exp(-NF(\mu^N)) \geq \bar{\mathbb{E}}_{\Pi^N}\left[\frac{1}{N}\sum_{i=1}^N \left(R(\bar{\mathbf{\Phi}}_i^N \| \Phi) + \frac{1}{2}\int_0^T |\psi_i^N(s)|^2 ds\right) + F(\bar{\mu}^N)\right] - \epsilon.$$

- Since F is bounded, there is a $C \in (0,\infty)$ such that

$$\sup_{N\in\mathbb{N}} \bar{\mathbb{E}}_{\Pi^N}\left(\frac{1}{N}\sum_{i=1}^N R(\bar{\mathbf{\Phi}}_i^N \| \Phi)\right) \leq C, \quad \sup_{N\in\mathbb{N}} \bar{\mathbb{E}}_{\Pi^N}\left(\frac{1}{2N}\sum_{i=1}^N \int_0^T |\psi_i^N(s)|^2 ds\right) \leq C.$$

- By a localization argument, we can assume that for every N

$$\frac{1}{2N}\sum_{i=1}^N \int_0^T |\psi_i^N(s)|^2 ds \leq C \quad a.s.$$

Consequences of Bounded Costs

Suppose

$$\sup_{N\in\mathbb{N}} \bar{\mathbb{E}}_{\Pi^N}\Big(\tfrac{1}{N}\sum_{i=1}^N R(\bar{\mathbf{\Phi}}_i^N\|\Phi)\Big)\le C_0,\ \sup_{N\in\mathbb{N}}\Big(\tfrac{1}{2N}\sum_{i=1}^N \int_0^T |\psi_i^N(s)|^2 ds\Big)\le C_0.$$

- Lemma 1 Let $\bar{\mathbb{Q}}_{\Pi^N}(t) = \mathcal{L}(\bar{X}^N(t))$. Then, there exists $C_T \in (0,\infty)$ s.t. for $t \in [0, T]$,
$$H_N(t) \doteq R(\bar{\mathbb{Q}}_{\Pi^N}(t)\|\Phi^N) \le C_T N \text{ for all } N \in \mathbb{N}.$$
- Let $V_i = \partial_i - \partial_{i+1}$ and a positive f on $\mathbb{R}^N$ that is continuously differentiable along $V_1, \ldots, V_N$, define
$$I_N(f) = 4D_N(\sqrt{f}) = \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{(V_i f(x))^2}{f(x)} \Phi^N(dx).$$
- Lemma 2 For $t \in [0, T]$ and $N \in \mathbb{N}$, $\bar{X}^N(t)$ has a density $\bar{p}_N(t,\cdot)$ w.r.t Φ^N which is continuously differentiable, once in time and twice along $V_1, \ldots, V_N$, and satisfies for some $C \in (0,\infty)$:
$$I_N\left(\frac{1}{T}\int_0^T \bar{p}_N(s,\cdot)ds\right) \le \frac{C}{N} \text{ for all } N \ge 1.$$

Consequences of Bounds on RE and Dirichlet Forms

- $\{\bar{\mu}^N\}$ is a tight sequence of Ω-valued random variables.

- $\{\nu^N\}$ is a tight sequence of $\mathcal{P}(\mathbb{R} \times S)$ valued r.v., where

$$\nu^N(dx\, d\theta) \doteq \sum_{i=1}^N \bar{\boldsymbol{\Phi}}_i^N(dx)\mathbb{I}_{(i/N,(i+1)/N]}(\theta)d\theta.$$

- Define u_N as

$$u^N(t,\theta) \doteq \sum_{i=1}^N \psi_i^N(t)\mathbb{I}_{((i-1)/N,i/N]}(\theta),\ (t,\theta) \in [0,T] \times S.$$

  Then $\{u^N\}$ is a sequence of r.v. with values in

$$\mathcal{S}_{C_0} \doteq \left\{ u \in L^2([0,T] \times S) : \int_{[0,T]\times S} |u(t,\theta)|^2 d\theta dt \le C_0 \right\}.$$

  With the weak topology on the Hilbert space $\mathcal{S}_{C_0}$ is compact and thus $\{u^N\}_{N\in\mathbb{N}}$ is a tight sequence of $\mathcal{S}_{C_0}$-valued random variables.

- Suppose $(\bar{\mu}^N, u^N, \nu^N)$ converges in distribution along a subsequence to $(\bar{\mu}, u, \nu)$. Then $\bar{\mu} \in \mathcal{M}_\infty(u,\nu)$, a.s.

Completing the Proof of Upper Bound.

$$
\begin{aligned}
&\liminf_{N\to\infty} -\frac{1}{N}\log \mathbb{E}\exp\left(-NF\left(\mu^N\right)\right)+\epsilon \\
&\geq \liminf_{N\to\infty} \bar{\mathbb{E}}_{\Pi^N}\left(F\left(\bar{\mu}^N\right)+\frac{1}{N}\sum_{i=1}^{N}\left(R(\bar{\boldsymbol{\Phi}}_i^N\|\Phi)+\frac{1}{2}\int_0^T|\psi_i^N(s)|^2ds\right)\right) \\
&= \liminf_{N\to\infty} \bar{\mathbb{E}}_{\Pi^N}\left(F\left(\bar{\mu}^N\right)+R(\nu^N\|\pi_0)+\frac{1}{2}\int_0^T\int_S|u_N(s,\theta)|^2dsd\theta\right) \\
&\geq \bar{\mathbb{E}}\left(F\left(\bar{\mu}\right)+R(\nu\|\pi_0)+\frac{1}{2}\int_0^T\int_S|u(s,\theta)|^2dsd\theta\right) \\
&\geq \bar{\mathbb{E}}\left[F(\bar{\mu})+I(\bar{\mu})\right] \quad \geq \inf_{\mu\in\Omega}\left[F(\mu)+I(\mu)\right],
\end{aligned}
$$

where the next to last inequality follows from $\bar{\mu}\in\mathcal{M}_\infty(u,\nu)$ a.s.

Laplace Lower Bound.

- Fix $F \in C_b(\Omega)$, and let $\epsilon > 0$. Choose $\bar{\mu}^* \in \Omega$ such that

$$F(\bar{\mu}^*) + I(\bar{\mu}^*) \leq \inf_{\mu \in \Omega} \{F(\mu) + I(\mu)\} + \epsilon < \infty,$$

- Choose $u^* \in L^2([0, T] \times S)$ and $\pi^* \in \mathcal{P}_*(\mathbb{R} \times S)$ such that $\bar{\mu}^* \in \mathcal{M}_\infty(u^*, \pi^*)$ and

$$I(\bar{\mu}^*) + \epsilon \geq \frac{1}{2}\left[\int_0^T \int_S |u^*(s, \theta)|^2 d\theta ds\right] + R(\pi^* \| \pi_0).$$

- Fix $\delta \in (0, 1)$ and let $u^{**} \in C^\infty$ be such that $\|u^{**} - u^*\|_2 \leq \frac{\delta}{2(1+\|u^*\|_2)}$.
- Lemma There is a unique $\bar{\mu}^{**} \in \Omega$ s.t. $\bar{\mu}^{**} \in \mathcal{M}(u^{**}, \pi^*)$. Furthermore, as $\delta \to 0$, $\bar{\mu}^{**} \equiv \bar{\mu}^{**}(\delta) \to \bar{\mu}^*$.

Control Synthesis.

- Let $\pi^*(dx, d\theta) = \pi_1^*(dx|\theta)d\theta$ and define

$$\bar{\Phi}_i^N(dx) \doteq N \int_{(i-1)/N}^{i/N} \pi_1^*(dx|\theta)d\theta, \quad 1 \leq i \leq N,$$

- Let $\bar{X}^N(0)$ be a $\mathbb{R}^N$-valued r.v. with distribution

$$\Pi^N(dx) \doteq \bar{\Phi}_1^N(dx_1) \dots \bar{\Phi}_N^N(dx_N).$$

- Define $\psi_i^N \in L^2([0, T] : \mathbb{R})$ as

$$\psi_i^N(t) \doteq \sum_{j=1}^{N} u^{**}\left(\frac{jT}{N}, \frac{i}{N}\right) \mathbb{I}_{(jT/N,(j+1)T/N]}(t), \ t \in [0, T].$$

Control Synthesis.

- Lemma Let $\bar{\mu}^N$ be constructed using Π^N and $\{\psi_i^N\}$. Then

$$\lim_{N\to\infty} \frac{1}{N}\int_0^T \sum_{i=1}^N |\psi_i^N(t)|^2 dt = \int_0^T \int_S |u^{**}(t,\theta)|^2 d\theta dt,$$

$$\frac{1}{N}\sum_{i=1}^N R(\bar{\mathbf{\Phi}}_i^N \| \Phi) \leq R(\pi^* \| \pi_0), \quad \text{for all } N \in \mathbb{N}$$

and $\bar{\mu}^N$ converges to $\bar{\mu}^{**}$ in distribution in Ω.

Completing the Proof of Lower Bound.

- Choose δ small s.t. $|F(\bar{\mu}^*) - F(\bar{\mu}^{**})| \leq \epsilon$.
- Then

$$\begin{aligned}
&\limsup_{N\to\infty} -\frac{1}{N}\log \mathbb{E}\exp\left(-NF\left(\mu^N\right)\right) \\
&\leq \limsup_{N\to\infty} \bar{\mathbb{E}}_{\Pi^N}\left(F\left(\bar{\mu}^N\right) + \frac{1}{N}\sum_{i=1}^{N}\left(R(\bar{\mathbf{\Phi}}_i^N\|\Phi) + \frac{1}{2}\int_0^T |\psi_i^N(s)|^2 ds\right)\right) \\
&\leq F(\bar{\mu}^{**}) + R(\pi^*\|\pi_0) + \frac{1}{2}\int_S\int_0^T |u^{**}(s,\theta)|^2 ds d\theta \\
&\leq F(\bar{\mu}^{*}) + R(\pi^*\|\pi_0) + \frac{1}{2}\int_S\int_0^T |u^{*}(s,\theta)|^2 ds d\theta + \epsilon + 2\delta \\
&\leq F(\bar{\mu}^*) + I(\bar{\mu}^*) + 2\epsilon + 2\delta \\
&\leq \inf_{\mu\in\Omega}\{F(\mu) + I(\mu)\} + 3\epsilon + 2\delta.
\end{aligned}$$