Volatility and arbitrage

Robert Fernholz
INTECH

Joint research with
Ioannis Karatzas and Johannes Ruf

University of Southern California
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Introduction

In a stock market, if there is "adequate volatility", then there is relative arbitrage. We shall investigate what "adequate volatility" might mean, when there is long-term arbitrage, and when there is arbitrage over arbitrarily short intervals.

The market

Suppose we have a market of stocks $X_1, \dots, X_n$ represented by positive continuous semimartingales that satisfy

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^{d} \xi_{i\nu}(t)\, dW_\nu(t),$$

for $i = 1, \dots, n$, where $d \geq n$, $(W_1, \dots, W_d)$ is a d-dimensional Brownian motion, and the processes γ_i and $\xi_{i\nu}$ are measurable, adapted to the Brownian filtration, and locally integrable or square-integrable. The process X_i represents the total capitalization of the ith company. The *market weights* are

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)},$$

for $i = 1, \dots, n$.

Covariance

The ijth *covariance process* σ_{ij} is defined for $i, j = 1, \ldots, n$ by

$$\sigma_{ij}(t) \triangleq \frac{d\langle \log X_i, \log X_j \rangle_t}{dt}$$
$$= \sum_{\nu=1}^{d} \xi_{i\nu}(t)\xi_{j\nu}(t), \quad \text{a.s.}$$

If the eigenvalues of the covariance matrix $\sigma(t) = (\sigma_{ij}(t))$ are uniformly bounded away from zero over an interval $[0, T]$, then the market is said to be *strongly nondegenerate* over the interval.

Portfolios

A *portfolio* π is defined by its *weight processes*, $\pi_1, \ldots, \pi_n$, which are bounded, measurable, adapted to the Brownian filtration, and add up to one. The portfolio *value process* Z_π represents the (positive) value of the portfolio and satisfies

$$d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\, d \log X_i(t) + \gamma_\pi^*(t)\, dt, \quad \text{a.s.},$$

where the *excess growth rate process* γ_π^* is defined by

$$\gamma_\pi^*(t) \triangleq \frac{1}{2}\Big(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t) \Big).$$

Due to the first equation, γ_π^* is effectively observable.

The market portfolio

The *market portfolio* μ is defined by the market weights $\mu_1, \ldots, \mu_n$, and

$$Z_\mu(t) = X_1(t) + \cdots + X_n(t), \quad \text{a.s.},$$

with appropriate initial conditions.

The ijth *relative covariance process* τ_{ij} is defined for $i, j = 1, \ldots, n$ by

$$\begin{aligned} \tau_{ij}(t) &\triangleq \frac{d\langle \log \mu_i, \log \mu_j \rangle_t}{dt} = \frac{d\langle \log(X_i/Z_\mu), \log(X_j/Z_\mu) \rangle_t}{dt} \\ &= \sigma_{ij}(t) - \sigma_{i\mu}(t) - \sigma_{j\mu}(t) + \sigma_{\mu\mu}(t), \quad \text{a.s.} \end{aligned}$$

Diverse markets

A market is *diverse* over the interval $[0, T]$ if there exists a $\delta > 0$ such that for $i = 1, \dots, n$,

$$\sup_{t \in [0,T]} \mu_i(t) < 1 - \delta, \quad \text{a.s.}$$

Lemma. If a market is strongly nondegenerate and diverse over $[0, T]$, then there exists $\varepsilon > 0$ such that for $i = 1, \dots, n$,

$$\inf_{t \in [0,T]} \tau_{ii}(t) > \varepsilon, \quad \text{a.s.}$$

Proof. (F (2002).) Let $x(t) = (\mu_1(t), \dots, \mu_i(t) - 1, \dots, \mu_n(t))$, so $\tau_{ii}(t) = x(t)\sigma(t)x^T(t) \geq c\|x(t)\|^2 > c(1 - \mu_i(t))^2 > c\delta^2$, a.s. □

Relative arbitrage

For $T > 0$, there is *relative arbitrage* versus the market on $[0, T]$ if there exists a portfolio π such that

$$\mathbb{P}\big[Z_\pi(T)/Z_\mu(T) \geq Z_\pi(0)/Z_\mu(0)\big] = 1,$$
$$\mathbb{P}\big[Z_\pi(T)/Z_\mu(T) > Z_\pi(0)/Z_\mu(0)\big] > 0.$$

It is *strong relative arbitrage* if

$$\mathbb{P}\big[Z_\pi(T)/Z_\mu(T) > Z_\pi(0)/Z_\mu(0)\big] = 1.$$

We are interested in conditions under which volatility produces relative arbitrage.

Functionally generated portfolios

Suppose that $\mathbf{S}$ is a positive C^2 function defined on a neighborhood of the open simplex

$$\Delta^n = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1, x_i > 0\}.$$

Then $\mathbf{S}$ *generates* a portfolio π such that

$$d \log \big(Z_\pi(t)/Z_\mu(t)\big) = d \log \mathbf{S}(\mu(t)) + d\Theta(t), \quad \text{a.s.},$$

for $t \in [0, T]$, where the *drift process* Θ is of bounded variation. The weights π_i and drift process Θ are determined by the partial derivatives of $\mathbf{S}$ and the covariance matrix of the market.
(F (2002).)

Relative variance and relative arbitrage

Proposition 1. If there exists an $\varepsilon > 0$ and a $k \in \{1, \ldots, n\}$ such that $\tau_{kk}(t) > \varepsilon$ for all $t \in [0, T]$, a.s., then there exists strong relative arbitrage versus the market over $[0, T]$.

Proof. (FKK (2005).) For $p > 1$, consider the function $\mathbf{S}(x) = x_k^p$, defined for $x \in \Delta^n$, the unit simplex in $\mathbb{R}^n$. The function $\mathbf{S}$ generates the portfolio π with weights

$$\pi_i(t) = \begin{cases} p - (p-1)\mu_i(t), & \text{for } i = k, \\ -(p-1)\mu_i(t), & \text{otherwise,} \end{cases}$$

and the value process Z_π satisfies

$$d \log \left(Z_\pi(t)/Z_\mu(t) \right) = d \log \mu_k^p(t) - \frac{p^2 - p}{2} \, \tau_{kk}(t) \, dt, \quad \text{a.s.}$$

Relative variance and relative arbitrage

Essentially, the portfolio π holds p dollars of X_k and $-(p-1)$ dollars of the market portfolio. We have

$$\begin{aligned}\log\big(Z_\pi(T)/Z_\mu(T)\big) &- \log\big(Z_\pi(0)/Z_\mu(0)\big)\\ &= \log\big(\mu_k^p(T)/\mu_k^p(0)\big) - \frac{p^2-p}{2}\int_0^T \tau_{kk}(t)\,dt\\ &\leq -p\log\mu_k(0) - \frac{(p^2-p)\varepsilon T}{2}, \quad \text{a.s.}\end{aligned}$$

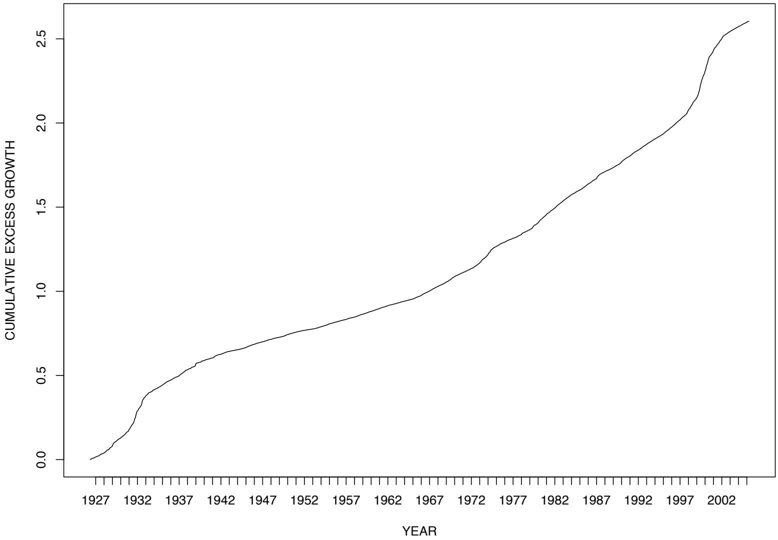
If p is large enough, then Z_π will underperform Z_μ, a.s. By shorting π and immersing it in a large amount of the market portfolio, we can construct a long-only portfolio that outperforms Z_μ, a.s., over $[0, T]$. □

Market excess growth

The market excess growth rate γ_μ^* measures the average relative volatility available in the market:

$$\begin{aligned}
\gamma_\mu^*(t) &= \frac{1}{2}\Big(\sum_{i=1}^{n} \mu_i(t)\sigma_{ii}(t) - \sigma_{\mu\mu}(t)\Big) \\
&= \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\Big(\sigma_{ii}(t) - 2\sigma_{i\mu}(t) + \sigma_{\mu\mu}(t)\Big) \\
&= \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t), \quad \text{a.s.}
\end{aligned}$$

Cumulative γ_μ^* for the U.S. market



Market entropy

The *entropy* function $\mathbf{S}$ is defined by

$$\mathbf{S}(x) \triangleq -\sum_{i=1}^{n} x_i \log x_i,$$

for $x \in \Delta^n$. The entropy function satisfies

$$0 \leq \mathbf{S}(x) \leq \log n$$

where the value 0 is attained only at the vertices of the simplex, and $\log n$ is attained only when all the x_i are equal to $1/n$. For a constant $c \geq 0$, we define the *generalized entropy function* by

$$\mathbf{S}_c(x) \triangleq \mathbf{S}(x) + c, \qquad \text{for } x \in \Delta^n.$$

Entropy-weighted portfolios

The generalized entropy function $\mathbf{S}_c$ generates the portfolio π with weights

$$\pi_i(t) = \frac{c - \log \mu_i(t)}{\mathbf{S}_c(\mu(t))} \mu_i(t),$$

and the value process Z_π of this *entropy-weighted* portfolio satisfies

$$d \log \big(Z_\pi(t)/Z_\mu(t)\big) = d \log \mathbf{S}_c(\mu(t)) + \frac{\gamma^*_\mu(t)}{\mathbf{S}_c(\mu(t))}\, dt, \quad \text{a.s.}$$

Long-term relative arbitrage

Proposition 2. Suppose that in a market defined for $t \geq 0$ there is an $\varepsilon > 0$ such that for all t, $\gamma_\mu^*(t) > \varepsilon$, a.s. Then for large enough T, there exists strong relative arbitrage versus the market on $[0, T]$.

Proof. For $c > 0$, consider the portfolio π generated by $\mathbf{S}_c$. Then

$$\begin{aligned}\log\big(Z_\pi(T)/Z_\mu(T)\big) &- \log\big(Z_\pi(0)/Z_\mu(0)\big)\\ &= \log\big(\mathbf{S}_c(\mu(T))/\mathbf{S}_c(\mu(0))\big) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}_c(\mu(t))}\,dt\\ &> \log\left(\frac{c}{c+\log n}\right) + \frac{\varepsilon T}{c+\log n}, \quad \text{a.s.}\end{aligned}$$

Hence, it is just a matter of choosing T large enough. □

Short-term relative arbitrage

It would perhaps be nice if $\gamma_\mu^*(t) > \varepsilon > 0$ implied short-term relative arbitrage, but this is not quite true. Instead:

Proposition 3. For $T > 0$, suppose that there exists an $\varepsilon > 0$ such that

$$\gamma_\mu^*(t) > \varepsilon, \quad \text{a.s.},$$

for all $t \in [0, T]$, and that for the entropy function $\mathbf{S}$,

$$\operatorname{ess\,inf}\big\{\mathbf{S}(\mu(t)) : t \in [0, T/2]\big\} \\ \leq \operatorname{ess\,inf}\big\{\mathbf{S}(\mu(t)) : t \in [T/2, T]\big\}.$$

Then there exists relative arbitrage versus the market on $[0, T]$.

Short-term relative arbitrage

Proof. Let

$$A = \operatorname{ess\ inf}\{\mathbf{S}(\mu(t)) : t \in [0, T/2]\}.$$

Since $\gamma_\mu^*(t) \geq \varepsilon > 0$ on $[0, T]$, a.s., not all the μ_i can be constantly equal to $1/n$, so

$$0 \leq A < \log n, \quad \text{a.s.}$$

Hence, we can choose $\delta > 0$ such that

$$A + 2\delta < \log n,$$

and

$$\mathbb{P}\Big[\inf_{t \in [0, T/2]} \mathbf{S}(\mu(t)) < A + \delta\Big] > 0.$$

Short-term relative arbitrage

Let us define the stopping time

$$\tau_1 = \inf \left\{ t \in [0, T/2] : \mathbf{S}(\mu(t)) \leq A + \delta \right\} \wedge T,$$

in which case

$$\mathbb{P}\left[\tau_1 \leq T/2\right] > 0.$$

We can now define a second stopping time

$$\tau_2 = \inf \left\{ t \in [\tau_1, T] : \mathbf{S}(\mu(t)) = A + 2\delta \right\} \wedge T,$$

and we have $\tau_1 \leq \tau_2$, a.s.

Short-term relative arbitrage

Now consider the generalized entropy function

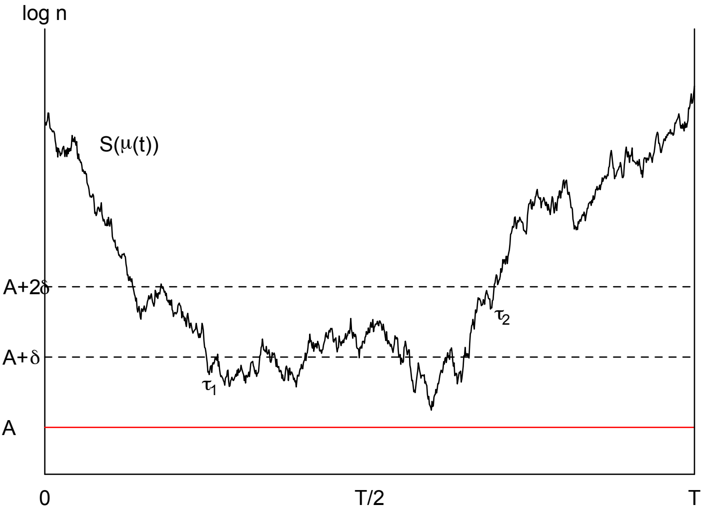
$$\mathbf{S}_\delta(x) \triangleq \mathbf{S}(x) + \delta,$$

for the same $\delta > 0$ as we chose above, so $\mathbf{S}_\delta(x) \geq \delta$. Let π be generated by $\mathbf{S}_\delta$, and we have

$$\begin{aligned}
&\log\big(Z_\pi(\tau_2)/Z_\mu(\tau_2)\big) - \log\big(Z_\pi(\tau_1)/Z_\mu(\tau_1)\big) \\
&\quad = \log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{\mathbf{S}_\delta(\mu(t))}\, dt, \quad \text{a.s.},
\end{aligned}$$

for the times τ_1 and τ_2.

Short-term relative arbitrage



Short-term relative arbitrage

Suppose that $\tau_1 \leq T/2$, so $\tau_1 < \tau_2$, a.s. There are two cases:

1. If $\tau_2 < T$, then

$$\begin{aligned} \log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) & \\ \geq \log(A + 3\delta) - \log(A + 2\delta) & \\ > 0, \quad \text{a.s.,} & \end{aligned}$$

and since
$$\int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{\mathbf{S}_\delta(\mu(t))}\, dt > 0, \quad \text{a.s.,}$$
we have

$$\log\left(Z_\pi(\tau_2)/Z_\mu(\tau_2)\right) - \log\left(Z_\pi(\tau_1)/Z_\mu(\tau_1)\right) > 0, \quad \text{a.s.}$$

Short-term relative arbitrage

2. If $\tau_2 = T$, then

$$A + \delta \leq \mathbf{S}_\delta(\mu(t)) < A + 3\delta, \quad \text{a.s.},$$

for $t \in [\tau_1, T]$, a.s., so

$$\log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{\mathbf{S}_\delta(\mu(t))}\, dt > \log \frac{A+\delta}{A+2\delta} + \frac{\varepsilon T}{2(A+3\delta)}, \quad \text{a.s.}$$

Again there are two cases:

Short-term relative arbitrage

1. If $A = 0$, let

$$\delta = \frac{\varepsilon T}{6 \log 2},$$

in which case,

$$\log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{\mathbf{S}_\delta(\mu(t))}\, dt$$
$$> \log \frac{A+\delta}{A+2\delta} + \frac{\varepsilon T}{2(A+3\delta)} = 0, \quad \text{a.s.},$$

so

$$\log \big(Z_\pi(\tau_2)/Z_\mu(\tau_2) \big) - \log \big(Z_\pi(\tau_1)/Z_\mu(\tau_1) \big) > 0, \quad \text{a.s.}$$

Short-term relative arbitrage

2. If $A > 0$, then

$$\lim_{\delta \downarrow 0} \left[\log \frac{A+\delta}{A+2\delta} + \frac{\varepsilon T}{2(A+3\delta)} \right] = \frac{\varepsilon T}{2A} > 0,$$

so for small enough $\delta > 0$

$$\log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{\mathbf{S}_\delta(\mu(t))}\, dt > \log \frac{A+\delta}{A+2\delta} + \frac{\varepsilon T}{2(A+3\delta)} > 0, \quad \text{a.s.},$$

and

$$\log \big(Z_\pi(\tau_2)/Z_\mu(\tau_2) \big) - \log \big(Z_\pi(\tau_1)/Z_\mu(\tau_1) \big) > 0, \quad \text{a.s.}$$

Short-term relative arbitrage

Now consider the portfolio η defined by:

1. For $t \in [0, \tau_1)$, $\eta(t) = \mu(t)$, the market portfolio.
2. For $t \in [\tau_1, \tau_2)$, $\eta(t) = \pi(t)$, the portfolio generated by $\mathbf{S}_\delta$ with δ chosen as in the two cases we considered.
3. For $t \in [\tau_2, T]$, $\eta(t) = \mu(t)$.

Short-term relative arbitrage

If $\tau_1 = T$, then $\eta(t) = \mu(t)$ for all $t \in [0, T]$, so

$$\log\left(Z_\eta(T)/Z_\mu(T)\right) = \log\left(Z_\eta(0)/Z_\mu(0)\right), \quad \text{a.s.}$$

If $\tau_1 \neq T$, then $\tau_1 \leq T/2$ and $\tau_1 < \tau_2$, a.s. By the construction of η, we have

$$\begin{aligned}\log\left(Z_\eta(T)/Z_\mu(T)\right) - \log\left(Z_\eta(0)/Z_\mu(0)\right) \\ &= \log\left(Z_\pi(\tau_2)/Z_\mu(\tau_2)\right) - \log\left(Z_\pi(\tau_1)/Z_\mu(\tau_1)\right) \\ &> 0, \quad \text{a.s.},\end{aligned}$$

with the inequality following from two the cases we considered.

Since $\mathbb{P}[\tau_1 \neq T] > 0$, there exists relative arbitrage on $[0, T]$. □

Adequate volatility

Corollary. Suppose that $\gamma^*_\mu(t) > \varepsilon > 0$, a.s., for $t \in [0, T]$, and that the market is strongly nondegenerate over that interval. Then there exists relative arbitrage versus the market on $[0, T]$.

Proof. There are two cases:

1. If the market is diverse over $[0, T/2]$, then Proposition 1 ensures short-term strong relative arbitrage.
2. If the market is not diverse over $[0, T/2]$, then $A = 0$ in Proposition 3, and short-term relative arbitrage follows. □

An example, with variations

Let $n = 3$, let $T > 0$, and let $0 < a < e^{-T/2}/9$. Suppose that (W, θ, B) is a 3-dimensional Brownian motion with the usual filtration $\mathcal{F}$. For $t \in [0, T]$ and for $i = 1, 2, 3$, define

$$X_i(t) = e^{W(t) - t/2}\Big(\frac{1}{3} + \varphi(t)e^{t/2}\cos\big(\theta(t) + (i-1)2\pi/3\big)\Big),$$

where φ is a martingale driven by B with $a < \varphi(t) < 3a$. Then the processes X_i are martingales, and it can be shown that $\gamma^*_\mu(t) > 3a^2/4$. Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by W, θ, and φ span $\mathbb{R}^3$, the covariance matrix is nonsingular. This market is not strongly nondegenerate.

An example, with variations

We define an $\mathcal{F}$-martingale ψ for $t \in [0, T]$ by

$$\psi(t) = \int_0^t \left(a^2 - \psi^2(s)\right) dB(s),$$

and we have

$$-a < \psi(t) < a, \quad \text{a.s.}$$

Then define φ for $t \in [0, T]$ by

$$\varphi(t) = 2a + \psi(t),$$

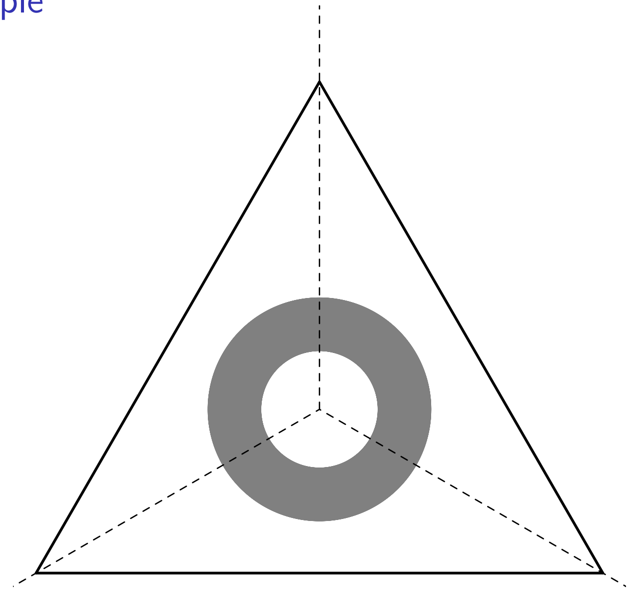
so

$$a < \varphi(t) < 3a, \quad \text{a.s.},$$

and

$$d\langle\varphi\rangle_t = d\langle\psi\rangle_t = \left(a^2 - \psi^2(t)\right)^2 dt, \quad \text{a.s.}$$

An example



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Variations

The weights $\mu_i(t)$ for the model

$$X_i(t) = e^{W(t)-t/2}\Big(\frac{1}{3} + \quad a \quad \cos\big(\theta(t) + (i-1)2\pi/3\big)\Big),$$

lie in a circle on the simplex Δ^3 centered at $(1/3, 1/3, 1/3)$, so

$$\mathbf{S}(\mu(t)) = \big(\mu_1^2(t) + \mu_2^2(t) + \mu_3^2(t)\big)^{1/2} = \text{ const.}$$

S generates a portfolio π with value function Z_π such that

$$\begin{aligned} d\log\big(Z_\pi(t)/Z_\mu(t)\big) &= d\log \mathbf{S}(\mu(t)) - \gamma_\pi^*(t)\,dt \\ &= -\gamma_\pi^*(t)\,dt, \quad \text{a.s.} \end{aligned}$$

Since $\gamma_\pi^*(t) > 0$, this produces *immediate relative arbitrage.*

Variations

Let $n = 3$, let $T > 0$, and let $0 < a < e^{-T/2}/9$. Suppose that (W, θ, B) is a 3-dimensional Brownian motion with the usual filtration $\mathcal{F}$. For $t \in [0, T]$ and for $i = 1, 2, 3$, define

$$X_i(t) = e^{W(t)-t/2}\Big(\frac{1}{3} + \varphi(t) \qquad \cos\big(\theta(t) + (i-1)2\pi/3\big)\Big),$$

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$$X_i(t) = \qquad \kappa(t)\Big(\frac{1}{3} + \varphi(t)e^{t/2}\cos\big(\theta(t) + (i-1)2\pi/3\big)\Big),$$

where φ is a martingale driven by B with $a < \varphi(t) < 3a$. Then ~~the processes X_i are martingales, and~~ it can be shown that $\gamma^*_\mu(t) > 3a^2/4$. ~~Since the price processes in this market are martingales,~~ relative arbitrage does not exist. Since the motions induced by $\kappa > 0$, θ, and φ span $\mathbb{R}^3$, the covariance matrix is nonsingular. This market is not strongly nondegenerate.

Volatility and arbitrage

Conclusion: $\gamma_\mu^*(t) > \varepsilon > 0$ will generate relative arbitrage, but not over arbitrarily short intervals.

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Thank you!