A Set-Valued Markov Chain Approach to Credit Default

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Mathematical Finance Colloquium, USC
March 9, 2020
HIGHLIGHTS

▶ General credit default model: consider *intergroup contagion* and *macroeconomic factors*

▶ A *set-valued* Markov chain (MC) for the default process $X$ show such an MC exists under given conditions

▶ *Explicit* pricing formulas of CDO spreads

▶ *Empirical studies* to showcase the theoretical results
OUTLINE

INTRODUCTION

A SET-VALUED APPROACH

MAIN THEORETICAL RESULTS

PRICING

NUMERICAL STUDIES
Motivation
**Motivation**

**Collateralized Debt Obligations**

Collateralized debt obligations (CDOs) are structured financial instruments that purchase and pool financial assets such as the riskier tranches of various mortgage-backed securities.

1. **Purchase**
   The CDO manager and securities firm select and purchase assets, such as some of the lower-rated tranches of mortgage-backed securities.

2. **Pool**
   The CDO manager and securities firm pool various assets in an attempt to get diversification benefits.

3. **CDO tranches**
   Similar to mortgage-backed securities, the CDO issues securities in tranches that vary based on their place in the cash flow waterfall.

*New pool of RMBS and other securities*

*First claim to cash flow from principal & interest payments...*
BASIC SETUP

- $N$ defaultable obligors (names): $O_i$
  where $i \in \mathbb{N} = \{1, 2, \cdots, N\}$

- stopping time $\nu_i$: default time of $O_i$

- $1 - R_i$: proportional nominal loss of $O_i$, where $R_i \in (0, 1)$

- loss process $L = (L_t)_{t \geq 0}$

\[
L_t := \sum_{i=1}^{N} (1 - R_i) \mathbb{1}_{\{\nu_i \leq t\}}
\]

A credit derivative is a c.c. with payoff depending on $L$.

- exogenous process $Y$: macroeconomic factors
  impact of $Y$ on credit default probabilities; see Bonfim (2009) and Chen (2010)
LITERATURE ON DEFAULT MODELS

1. Structure models
   Merton (1974), Black and Cox (1976), ...
   asset value < debt (barrier) ⇒ default

2. Copula models
   Li (2000), Hull and White (2004), ...
   copula to model joint distribution of defaults

3. Intensity-based models
LITERATURE ON INTENSITY-BASED MODELS

Two dominating approaches:

1. **bottom-up** approach
   specify the intensity process $\lambda_i$ for each name $O_i$ s.t.
   \[
   \left( 1_{\{ \nu_i \leq t \}} - \int_0^t \lambda_i(s) ds \right)_{t \geq 0} \text{ is a martingale}
   \]

2. **top-down** approach
   Errais et al. (2007, 2010), Giesecke et al. (2011), Cont and Minca (2013), ...
   specify the intensity process $\lambda_L$ for the aggregate loss $L$ s.t.
   \[
   \left( L_t - \int_0^t \lambda_L(s) ds \right)_{t \geq 0} \text{ is a martingale}
   \]
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Definition 1

\[ X = (X_t)_{t \geq 0} \] denotes the default process, where \( X_t \) is the set of names that have defaulted by time \( t \).

\( X \) is a set-value process taking values in subsets of \( \mathbb{N} \).

\( |X_t| \): number of elements in set \( X_t \) (cardinality).

Example. \( O_1 \) defaults at \( t = 0.1 \), \( O_5 \) at \( t = 0.5 \), and \( O_9 \) at \( t = 1 \); then \( X_1 = \{1, 5, 9\} \) and \( |X_1| = 3 \).

Standing Assumptions

(i) No more than one default occurs at the same time
(ii) Obligors do not recover after the default

\[ \mathbb{P}(|X_{t+\Delta t}|-|X_t| > 1) = o(\Delta t) \] and \( \tau_1 < \cdots < \tau_i < \cdots < \tau_N \)

\[ X \] is a non-decreasing process, i.e., \( X_s \subseteq X_t \) for all \( 0 \leq s < t \)

The process \( |X| \) jumps up by size 1 at default time \( \tau_i, i \in \mathbb{N} \)
Filtrations

- \( \mathbb{F}^Y = (\mathcal{F}_t^Y)_{t \geq 0} \): generated by the macroeconomic process \( Y \)
- \( \mathbb{F}^X = (\mathcal{F}_t^X)_{t \geq 0} \): generated by the default process \( X \)
- \( \mathbb{N} \): all the subsets of \( \mathbb{N} \), i.e., \( \mathbb{N} = 2^{\mathbb{N}} \) (\( \mathbb{N} = \{1, 2, \ldots, N\} \))

Definition 2

A continuous time \( \mathbb{N} \)-valued stochastic process \( X = (X_t)_{t \geq 0} \) is called an \( \mathbb{F}^Y \)-conditional Markov chain if, for all \( 0 \leq s \leq t \) and \( F \in \mathbb{N} \), the following condition holds:

\[
\mathbb{P} \left( X_t = F \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y \right) = \mathbb{P} \left( X_t = F \mid \sigma(X_s) \vee \mathcal{F}_s^Y \right), \quad \mathbb{P}\text{-a.s.}
\]

Conditioning on \( Y \), (the default process) \( X \) is a Markov chain.
Definition 3

A family of $\mathcal{F}^Y$-adapted processes $\Lambda = (\Lambda_{EF}(t))_{t \geq 0}$ or $\lambda = (\lambda_{EF}(t))_{t \geq 0}$, where $E \subseteq F \in \mathbb{N}$, is called the default intensity family of an $\mathbb{N}$-valued process $X = (X_t)_{t \geq 0}$, if the process $X_F = (X_F(t))_{t \geq 0}$, defined by

$$X_F(t) := 1_F(X_t) - \sum_{E \subseteq F} \int_0^t 1_{\{X_s = E\}} d\Lambda_{EF}(s)$$

is a martingale for all $F \in \mathbb{N}$ with respect to the filtration $(\mathcal{F}^X_t \vee \mathcal{F}^Y_t)_{t \geq 0}$, where

$$\Lambda_{EF}(t) := \int_0^t \lambda_{EF}(s) ds$$

Think: $X_s = E \rightarrow X_t = F$, where $0 \leq s \leq t$ and $E \subseteq F$
**Remarks**

- The intensity family $\Lambda$ or $\lambda$ in Definition 3 plays an important role in the **compensator** of the default process $X$.
- $\lambda_{EF}(t)$ represents the **conditional default rate** at time $t$ when obligors in set $E$ have already defaulted.
- We have $\lambda_{EF}(t) = 0$, whenever $F \neq E \cup \{i\}, i \in E^c$.

... wait a second. So far the framework really looks like the **Markov chain** models, see Bielecki et al. (2011)

**Key Difference**

We show that for suitable $\Lambda$ (or $\lambda$) and $Y$, there exists an $\mathcal{F}^Y$-conditional set-valued Markov chain $X$ taking values in $\mathbb{N}$. In comparison, the existing works usually begin with such a Markov chain.

**Benefit:** apply the market prices/spreads to recover default intensities
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**Existence Assumptions**

**Assumption 4**

\[ M = (M_{EF}(t))_{t \geq 0} \] is a family of Poisson processes with intensity 1, where \( E \subseteq F \). \( M \) and \( Y \) are independent.

**Assumption 5**

The intensity family \( \lambda = (\lambda_{EF}(t))_{t \geq 0} \) satisfies

- \( \lambda_{EF}(t) = 0 \), if \( E \neq F \) or \( F \neq E \cup \{i\} \), where \( i \in E^c \).
- \( \lambda_E(t) := -\lambda_{EE}(t) = \sum_{E \neq F} \lambda_{EF}(t) \).
- \( \lambda_{EF}(t) \geq 0 \) for all \( F = E \cup \{i\} \), where \( i \in E^c \).
- \( \lim_{t \to +\infty} \int_0^t \lambda_{EF}(s)ds = +\infty \) for all \( F = E \cup \{i\} \), where \( i \in E^c \).

**Note.** We can formulate Assumption 5 using \( \Lambda \).
Existence result

Theorem 6

Let Assumptions 4 and 5 hold, there exists an $\mathbb{F}^Y$-conditional Markov chain $X$ with intensity family $\lambda$ and $X_0 = \emptyset$.

Model Flexibility:

- $Y$ is arbitrary
- Minimum assumptions on the default intensity family $\lambda$
DYNAMICS OF X

\[ F \setminus E = \{ x : x \in F \text{ and } x \not\in E \}, \text{ where } E \subset F \text{ and } |F \setminus E| = n \]

\[ \Pi(F \setminus E) \] denotes the set of all the permutations of \( F \setminus E \)

\[ \forall \pi \in \Pi(F \setminus E), \text{ define a sequence of sets } (F^\pi_k)_{k=0,1,\ldots,n} \text{ by} \]

\[ F^\pi_0 := E \quad \text{and} \quad F^\pi_k := F^\pi_{k-1} \cup \{\pi_k\}, \quad k = 1, 2, \ldots, n \]

**Remark.** \( E \to F^\pi_1 \to F^\pi_2 \to \cdots \to F^\pi_n = F \) is a default path

Example. Let \( E = \{1, 2\} \) and \( F = \{1, 2, 3, 4\} \). Suppose \( X_s = E \) and \( X_t = F \), where \( s < t \). Since \( \Pi(F \setminus E) = \{(3, 4), (4, 3)\} \), from \( s \) to \( t \), obligors \( O_3 \) and \( O_4 \) have defaulted, and the path is

\[ F^\pi_1 = \{1, 2\} \to F^\pi_1 = \{1, 2, 3\} \to F^\pi_1 = \{1, 2, 3, 4\} \]

or \( F^\pi_2 = \{1, 2\} \to F^\pi_2 = \{1, 2, 4\} \to F^\pi_2 = \{1, 2, 3, 4\} \)
Theorem 7

Let Assumptions 4 and 5 hold. ∀ s ≤ t and F ∈ N, we have, for any bounded $F_t^\gamma$-measurable random variable $\xi$, that

$$
\mathbb{E} \left[ 1_{\{X_t=F\}} \cdot \xi \mid F_s^X \vee F_s^\gamma \right] = \sum_{E \subseteq F} 1_{\{X_s=E\}} \cdot \mathbb{E} \left[ \xi \ G(s, t; E, F) \mid F_s^\gamma \right]
$$

$$
H_0(s, t; E) := e^{-\int_s^t \lambda_E(u) du}, \quad \text{with} \quad \lambda_E(t) = -\lambda_{EE}(t) = \sum_{E \neq F} \lambda_{EF}(t)
$$

$$
H_{k+1}(s, t; \cdots) := \int_s^t \lambda_{F_k^\pi F_{k+1}^\pi}(v) \cdot e^{-\int_v^t \lambda_{F_k^\pi}(u) du} \cdot H_k(s, v; \cdots) dv
$$

$$
G(s, t; E, F) := \begin{cases} 
H_0(s, t; E), & \text{if } E = F \\
\sum_{\pi \in \Pi(F \setminus E)} H_{|F \setminus E|}(s, t; F_{\pi 0}, \cdots, F_{|F \setminus E|}^\pi), & \text{if } E \subset F 
\end{cases}
$$
**Remarks**

Given $X_t = F$, $X_s = E = F$ or $E \subset F$, explaining $\sum_{E \subset F} 1\{X_s = E\}$

$H_k(s, t; F_0^\pi, F_1^\pi, \cdots, F_k^\pi)$: probability that $X$ evolves from $X_s = E = F_0^\pi$ to $X_t = F_k^\pi$ (with $k$ defaults) in a particular path

Hence, $G(s, t; E, F)$ captures exactly the transition probability of $X$ from $X_s = E$ to $X_s = F$

**Remark.** If $N = 1$ or 2, we can obtain very simplified results on $\mathbb{P} \left[ X_t = F \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y \right]$

$\Rightarrow$ potential applications to the FTD (first-to-default baskets), where $N = 5$

If $N$ is large (e.g., $N = 125$ for iTraxx), the computations are intensity due to the involvement of permutations.
Assumption 8

Let constants $\beta_i, \rho_{ji} > 0$ and function $h(\cdot)$ be positive with $h(0) = 1$. We define, for all $E \in \mathbb{N}$ and $i \in E^c$, that

$$L_E(i) := \begin{cases} h(|E|) \cdot \sum_{j \in E} \rho_{ji}, & \text{if } E \neq \emptyset \\ \beta_i, & \text{if } E = \emptyset \end{cases}$$

and

$$\overline{L}_E := \sum_{i \in E^c} L_E(i), \quad \text{with} \quad \overline{L}_\emptyset := 0$$

Let $\Phi(\cdot, \cdot)$ be a positive functional mapping from $[0, \infty) \times \mathbb{R}^d$ to $\mathbb{R}^+$. The intensity family $\lambda = (\lambda_{EF}(t))_{t \geq 0}$ is given by

$$\lambda_{EF}(t) = \begin{cases} \Phi(t, Y_t) \cdot L_E(i), & \text{if } F = E \cup \{i\} \text{ and } i \in E^c \\ -\Phi(t, Y_t) \cdot \overline{L}_E, & \text{if } E = F \\ 0, & \text{otherwise} \end{cases}$$
MODEL EXPLANATIONS

- $\beta_i$: base default intensity of $O_i$ (no contagion)
- $\rho_{ji}$: individual contagion rate of $O_j$ on $O_i$
  Recall $j \in E$ (defaulted set) and $i \in E^c$ (surviving set)
- $h : \{0, 1, \cdots, N\} \rightarrow \mathbb{R}^+$: impact of default magnitude
- $L_E(i)$: intergroup contagion effect of $E$ on obligor $O_i$
- $\overline{L}_E$: aggregate impact of defaulted obligors in $E$ on all survivors in $E^c$
- $\Phi$: contagion effect of macroeconomic factors

**Note.** With Assumption 8 on the intensity family $\lambda$, we can further reduce the results of Theorem 7 (conditional probability and expectation of $X$).
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<th>Pricing</th>
<th>Numerical Studies</th>
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<tr>
<td><strong>Numerical Studies</strong></td>
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Assumption 9

(i) $\Phi(t, y) = y$ for all $t \geq 0$.

(ii) The macroeconomic factor process $Y$ is given by

$$dY_t = \kappa(\theta - Y_t)dt + \sigma \sqrt{Y_t}dW_t + dJ_t,$$

with $Y_0 = y_0$

$\kappa, \theta, \sigma > 0$, $W$ B.M., $J$ compound Poisson (primary parameter $l$ and secondary exponential with mean $\mu$). $W$ and $J$ are independent.

Duffie and Garleanu (2001) and Mortensen (2006):

$$\mathbb{E}\left[e^{-g \int_0^t Y_s ds}\right] = e^{A(g,0,t)+y_0\cdot B(g,0,t)}, \quad g > 0$$
Consider the **proportion** version of the loss process

\[ L_t = \frac{R_X(t)}{N} = \sum_{i \in X_t} (1 - R_i), \quad t \geq 0 \]

Attach points are \( 0 < \cdots < p_K \leq 1 \) and tranche \( i \) is \([p_{i-1}, p_i]\). The accumulated loss of tranche \( i \) is defined by

\[
L^{(i)}(X_t) := (L_t - p_{i-1})^+ - (L_t - p_i)^+
\]

- (Default Leg) The protection seller covers \( L^{(i)}(X_t) \).
- (Premium Leg) The protection buyer pays **upfront** fees \( u^{(i)} \Delta p_i = u^{(i)} \times (p_i - p_{i-1}) \) at inception and periodic premiums or spreads \( s^{(i)}(\Delta p_i - L^{(i)}(X_{t_{k-1}})) \Delta_k \) at each payment time \( t_k, \) where \( k = 1, \cdots, m \). \( (\Delta_k = 1/4 \) quarterly payments and \( m \) is the term.)
SPREADS

Proposition 10

\[
\begin{align*}
  s^{(i)} &= \sum_{k=1}^{m} e^{-rt_k} \mathbb{E} \left[ L^{(i)}(X_{t_k}) - L^{(i)}(X_{t_{k-1}}) \right] - u^{(i)} \Delta p_i \\
  &= \sum_{k=1}^{m} e^{-rt_k} \left( \Delta p_i - \mathbb{E}[L^{(i)}(X_{t_{k-1}})] \right) \Delta k
\end{align*}
\]

where \( \mathbb{E} \) denotes expectation under the risk neutral probability.

\[
\mathbb{E}[L^{(i)}(X_{t_k})] = \sum_{n=0}^{N} \sum_{F \in \mathcal{A}(n)} \sum_{\pi \in \Pi(F \setminus \emptyset)} \sum_{j=0}^{n} L^{(i)}(F) \hat{\mathcal{L}}_{\pi}^\pi(n) \alpha_j^{(n)}(\pi) \\
  \cdot \mathbb{E} \left[ e^{-\mathcal{L}_{F_j}^\pi \int_0^{t_k} \Phi(u,Y_u) \, du} \right]
\]
**Example I**

**Homogeneous Contagion Model (HCM)**

Let (intensity) Assumption 8 hold. We assume $\rho_{ij} = \rho$ for all $i \neq j$, $\Phi(t, y) = y$ and $h(n) = e^{-\delta n}$, where $\delta$ is a constant.

**Proposition 11**

Let Assumptions 4 and 9 hold. Under the HCM, we have

$$
\mathbb{E} \left[ L^{(i)}(X_{t_k}) \right] = \sum_{j=0}^{N-1} \Gamma_{j}^{(i)} \cdot \exp \left( A(a_j, 0, t_k) + y_0 B(a_j, 0, t_k) \right) + 1
$$

We can also compute $\mathbb{P}(|X_t| = n)$ explicitly.

**Note.** $\Gamma_j$ and $a_j$ are explicitly given.
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A set-valued approach

Main theoretical results

Pricing

Numerical studies
PART I. TOY EXAMPLES

- Number of obligors: $N = 125$
- Risk-free interest rate: $r = 5\%$
- Payment frequency: $\Delta = 1/4$ (quarterly)
- Recovery rate: $R_i \equiv R = 40\%$
- Process $Y$ (taken from Duffie and Garleanu (2001)):
  \[ y_0 = 1, \, \kappa = 0.6, \, \theta = 0.02, \, \sigma = 0.141, \, l = 0.2, \text{ and } \mu = 0.1 \]
- HCM parameters:
  \[ \rho = 0.05, \, \delta = -0.008, \text{ and } a_0 = 0.35 \]
Table 1: 5-year CDO Tranche Spreads under HCM and NCM

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<tr>
<th>Tranches</th>
<th>HCM Spread (bp)</th>
<th>NCM Spread (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 3%]</td>
<td>1502</td>
<td>918</td>
</tr>
<tr>
<td>[3%, 6%]</td>
<td>1240</td>
<td>590</td>
</tr>
<tr>
<td>[6%, 9%]</td>
<td>1095</td>
<td>511</td>
</tr>
<tr>
<td>[9%, 12%]</td>
<td>977</td>
<td>435</td>
</tr>
<tr>
<td>[12%, 22%]</td>
<td>839</td>
<td>359</td>
</tr>
<tr>
<td>[22%, 60%]</td>
<td>619</td>
<td>283</td>
</tr>
</tbody>
</table>

Note. NCM stands for Near-neighbour Contagion Model, where each obligor $O_i$ only impacts its two neighbors $O_{i-1}$ and $O_{i+1}$.
Table 2: Attachment and Detachment Time under HCM

<table>
<thead>
<tr>
<th>Tranches</th>
<th>Detachment #</th>
<th>Attachment t</th>
<th>Detachment t</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 3%]</td>
<td>7</td>
<td>0.5</td>
<td>1.25</td>
</tr>
<tr>
<td>[3%, 6%]</td>
<td>13</td>
<td>1.25</td>
<td>1.75</td>
</tr>
<tr>
<td>[6%, 9%]</td>
<td>19</td>
<td>1.75</td>
<td>2.25</td>
</tr>
<tr>
<td>[9%, 12%]</td>
<td>25</td>
<td>2.25</td>
<td>3</td>
</tr>
<tr>
<td>[12%, 22%]</td>
<td>46</td>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>[22%, 60%]</td>
<td>125</td>
<td>11</td>
<td>294</td>
</tr>
</tbody>
</table>

Note. $t$ is in unit of years.

$3\% \times N(125)/60\% = 6.25 \Rightarrow [0, 3\%]$ detach at the 7th default
Figure 1: Sensitivity Analysis of CDO Tranche Spreads under HCM
Recall $h(n) = e^{-\delta n}$ measures the magnitude
In addition, we have also considered recovery rate $R$, number of payments $m$, mean-reversion speak $\kappa$ of $Y$.

► The CDO tranche spreads are very sensitive to all factors considered here, except for the macroeconomy volatility $\sigma$.

► Among all six factors considered, only the default contagion rate $\rho$ is positively related with respect to the tranche spreads, while the rest shows negative relation.

► The tranche spreads are extremely elastic to the default contagion rate $\rho$ and contagion recovery rate $\delta$. One can interpret $\delta$ as the government intervention or self recovery rate of the group. The equity tranche is less sensitive to $\delta$ comparing with other tranches since it mainly reflects idiosyncratic risk.
PART II. MARKET CALIBRATION

- Data: 5-year CDX North American Investment Grade (5Y CDX.NA.IG) from Seo and Wachter (2018)
- Attachment points: 0, 3, 7, 10, 15, 30 (in percentage)
- Full sample: 10/05 - 9/08; Pre-crisis sample: 10/05 - 9/07; Post-crisis sample: 10/07 - 9/08
- Goal: estimate $\mathbf{x} = (a_0, \kappa, \theta, \sigma, \mu, l, \delta, \rho, y_0)$
- Best fit $\hat{x}$:

$$\min_{\mathbf{x}} \sum_{i=1}^{6} \left( \frac{\text{Tranche } i^{\text{model}} - \text{Tranche } i^{\text{market}}}{\text{Tranche } i^{\text{market}}} \right)^2$$
Table 3: Calibrated Parameters

<table>
<thead>
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<th>Parameters</th>
<th>Full</th>
<th>Pre-crisis</th>
<th>Post-crisis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>1.9762</td>
<td>1.1985</td>
<td>1.8978</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.5626</td>
<td>0.5631</td>
<td>0.3619</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.4428</td>
<td>0.1765</td>
<td>0.6893</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.1197</td>
<td>0.0743</td>
<td>0.1984</td>
</tr>
<tr>
<td>$\mu$</td>
<td>2.5805</td>
<td>1.8237</td>
<td>3.0000</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.5138</td>
<td>2.0000</td>
<td>0.2537</td>
</tr>
<tr>
<td>$\delta$</td>
<td>-0.0098</td>
<td>-0.0269</td>
<td>0.0079</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0025</td>
<td>0.0014</td>
<td>0.0039</td>
</tr>
<tr>
<td>$y_0$</td>
<td>1.8460</td>
<td>0.9974</td>
<td>2.1535</td>
</tr>
</tbody>
</table>

Note. $a_0 = \sum_{j=1}^{N} \beta_i$ (aggregate base default rates).

$h(n) = e^{-\delta n}$, $\delta < 0$ (resp. $\delta < 0$) implies positive (negative) effect on credit spreads.

$\rho$ (default intensity) almost tripled from 0.0014 to 0.0039.
Table 4: Calibration of 5Y CDX.NA.IG Tranches and Index

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Full Data</th>
<th>Full Model</th>
<th>Pre-crisis Data</th>
<th>Pre-crisis Model</th>
<th>Post-crisis Data</th>
<th>Post-crisis Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 3%]</td>
<td>39</td>
<td>26</td>
<td>31</td>
<td>17</td>
<td>54</td>
<td>37</td>
</tr>
<tr>
<td>[3%, 7%]</td>
<td>238</td>
<td>222</td>
<td>108</td>
<td>88</td>
<td>498</td>
<td>498</td>
</tr>
<tr>
<td>[7%, 10%]</td>
<td>102</td>
<td>96</td>
<td>25</td>
<td>26</td>
<td>255</td>
<td>229</td>
</tr>
<tr>
<td>[10%, 15%]</td>
<td>54</td>
<td>56</td>
<td>12</td>
<td>13</td>
<td>136</td>
<td>145</td>
</tr>
<tr>
<td>[15%, 30%]</td>
<td>27</td>
<td>26</td>
<td>6</td>
<td>6</td>
<td>69</td>
<td>65</td>
</tr>
<tr>
<td>[30%, 100%]</td>
<td>NA</td>
<td>11</td>
<td>NA</td>
<td>1</td>
<td>NA</td>
<td>27</td>
</tr>
<tr>
<td>Index</td>
<td>67</td>
<td>87</td>
<td>42</td>
<td>55</td>
<td>116</td>
<td>142</td>
</tr>
</tbody>
</table>


Table 5: Implied Default Contagion Rate $\rho$

<table>
<thead>
<tr>
<th></th>
<th>Full</th>
<th>Pre-crisis</th>
<th>Post-crisis</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 3%]</td>
<td>0.116%</td>
<td>0.045%</td>
<td>0.177%</td>
</tr>
<tr>
<td>[3%, 7%]</td>
<td>0.086%</td>
<td>0.035%</td>
<td>0.136%</td>
</tr>
<tr>
<td>[7%, 10%]</td>
<td>0.083%</td>
<td>0.033%</td>
<td>0.129%</td>
</tr>
<tr>
<td>[10%, 15%]</td>
<td>0.081%</td>
<td>0.033%</td>
<td>0.112%</td>
</tr>
<tr>
<td>[15%, 30%]</td>
<td>0.030%</td>
<td>0.035%</td>
<td>0.062%</td>
</tr>
</tbody>
</table>

implied $\rho$: model = data

implied default contagion rate smile
REFERENCES I


REFERENCES II


**REFERENCES III**


Chen, D.F., Deng, J., Feng, J.F. and Zou, B.
A Set-Valued Markov Chain Approach to Credit Default
Quantitative Finance (2020)
https://doi.org/10.1080/14697688.2019.1693053

**THANK YOU!**