Robust Quadratic Hedging via G-Expectation

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Introduction

- We consider the problem of mean-variance hedging in the context of volatility uncertainty, within the G-expectation framework.
- This talk is based on Biagini, F. , Mancin, J. , Meyer-Brandis, T. , *Robust Mean-Variance Hedging via G-Expectation*, Preprint LMU, 2016.

Preliminaries (Peng [2])

Let Ω be a given set and $\mathcal{H}$ be a vector lattice of real functions defined on Ω, i.e. a linear space containing 1 such that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$.

Definition

A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E}\colon \mathcal{H} \to \mathbb{R}$ satisfying the following properties

1. **Monotonicity:** If $X, Y \in \mathcal{H}$ and $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
2. **Constant preserving:** For all $c \in \mathbb{R}$ we have $\mathbb{E}[c] = c$.
3. **Sub-additivity:** For all $X, Y \in \mathcal{H}$ we have $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$.
4. **Positive homogeneity:** For all $X \in \mathcal{H}$ we have $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called *a sublinear expectation space*.

We consider a space $\mathcal{H}$ of random variables having the following property: if $X_i \in \mathcal{H},\ i = 1, \dots n$ then

$$\phi(X_1, \dots, X_n) \in \mathcal{H}, \quad \forall \phi \in C_{b,Lip}(\mathbb{R}^n),$$

where $C_{b,Lip}(\mathbb{R}^n)$ is the space of all bounded Lipschitz continuous functions on $\mathbb{R}^n$.

Definition

An m-dimensional random vector $Y = (Y_1, \dots, Y_m)$ is said to be independent of an n-dimensional random vector $X = (X_1, \dots, X_n)$ if for every $\phi \in C_{b,Lip}(R^n \times R^m)$

$$\mathbb{E}[\phi(X, Y)] = \mathbb{E}[\mathbb{E}[\phi(x, Y)]_{x=X}].$$

If $n = m$, we say that X and Y are identically distributed ($X \sim Y$), if for each $\phi \in C_{b,Lip}(\mathbb{R}^n)$

$$\mathbb{E}[\phi(X)] = \mathbb{E}[\phi(Y)].$$

Definition

A random variable X on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G-normal distributed if for any $a, b \geq 0$

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}X,$$

where $\bar{X}$ is an independent copy of X. Such X is symmetric, i.e. $\mathbb{E}(X) = \mathbb{E}(-X) = 0$.

The letter G denotes the function

$$G(y) := \frac{1}{2}\mathbb{E}(yX^2) : \mathbb{R} \mapsto \mathbb{R}.$$

We have the following identity

$$G(y) = \frac{1}{2}\overline{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-,$$

with $\overline{\sigma}^2 := \mathbb{E}(X^2)$ and $\underline{\sigma}^2 := -\mathbb{E}(-X^2)$.

Definition

A stochastic process $B = (B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G-Brownian motion if it satisfies the following conditions

1. $B_0 = 0$,
2. $B_t \in \mathcal{H}$ for each $t \geq 0$.
3. For each $t, s \geq 0$ the increment $B_{t+s} - B_t$ is independent of $(B_{t_1}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 < \ldots < t_n \leq t$.
4. $(B_{t+s} - B_t)s^{-1/2}$ is G-normally distributed.

- It is possible to choose a sub-linear space such that the canonical process is a G-Brownian motion. In this case the corresponding **sub-linear expectation** $\mathbb{E}_G$ is called G-**expectation**.

- Following Peng [3] and Denis, Hu, and Peng [1], we introduce the following notation: for each $t \in [0, \infty)$
    1. $\Omega = C_0(\mathbb{R}_+, \mathbb{R})$, $\Omega_t := \{\omega_{.\wedge t} : \omega \in \Omega\}$, $\mathcal{F}_t := \mathcal{B}(\Omega_t)$
    2. $L_{ip}(\Omega_t) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) | n \in \mathbb{N}, t_1, \dots, t_n \in [0, t], \varphi \in C_{b,Lip}(\mathbb{R}^n)\}$
    3. For $p \geq 1$, $L^p_G(\mathcal{F}_T)$ is the completion of $L_{ip}(\Omega_T)$ under the norm $\|\xi\|_p := \mathbb{E}_G\left[|\xi|^p\right]^{\frac{1}{p}}$.
    4. for $p \geq 1$, $M^p_G(0, T)$ is the completion of the set of elementary processes of the form

$$\eta(t) = \sum_{i=1}^{n-1} \xi_i I_{[t_i, t_{i+1})}(s),$$

where $0 \leq t_1 < t_2 < \ldots < t_n \leq T,\ n \geq 1$ and $\xi_i \in Lip(\Omega_{t_i})$ under the norm

$$\|\eta\|_{M^p_G(0,T)} := \mathbb{E}_G[\int_0^T |\eta(s)|^p ds]^{1/p}.$$

Definition

Consider

$$X = \phi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}), \quad \phi \in C_{b,Lip}(\mathbb{R}^{d \times n})$$

for $0 \leq t_1 < \ldots < t_n < \infty$. We define the conditional G-expectation under $\mathcal{F}_{t_1}$ as

$$\mathbb{E}_G[X|\mathcal{F}_{t_1}] := \psi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_j} - B_{t_{j-1}}),$$

where

$$\psi(x) := \mathbb{E}_G[\phi(x, B_{t_{j+1}} - B_{t_j}, \ldots, B_{t_n} - B_{t_{n-1}})].$$

Theorem (Denis, Hu, and Peng [1], Theorem 52 and 54)

Let $(\tilde{\Omega}, \mathcal{G}, \mathbb{P}_0)$ be a probability space carrying a standard Brownian motion W with respect to its natural filtration $\mathbb{G}$. Let $\Theta = [\underline{\sigma}, \overline{\sigma}]$ and denote by $\mathcal{A}^{\Theta}_{0,\infty}$ the set of all Θ-valued $\mathbb{G}$-adapted processes on $[0, \infty)$. For each $\theta \in \mathcal{A}^{\Theta}_{0,\infty}$ define $\mathbb{P}^{\theta}$ as the law of a stochastic integral $\int_0^{\cdot} \theta_s dW_s$ on the canonical space $\Omega = C_0(\mathbb{R}_+, \mathbb{R})$. We introduce the sets

$$\mathcal{P}_1 := \{\mathbb{P}^{\theta} \colon \theta \in \mathcal{A}^{\Theta}_{0,\infty}\}, \quad and \quad \mathcal{P} := \overline{\mathcal{P}_1}, \tag{2.1}$$

where the closure is taken in the weak topology. Then we have the representation

$$\mathbb{E}_G[X] = \sup_{\mathbb{P} \in \mathcal{P}_1} \mathbb{E}^{\mathbb{P}}[X] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[X], \quad for\ each\ X \in L^1_G(\Omega). \tag{2.2}$$

- Similarly an analogous representation holds for the G-conditional expectation.

Proposition (Soner, Touzi, and Zhang [5], Proposition 3.4)

Let $\mathcal{Q}(t,\mathbb{P}) := \{\mathbb{P}' \in \mathcal{Q} \colon \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_t\}$, *where* $\mathcal{Q} = \mathcal{P}$ *or* $\mathcal{P}_1$. *Then for any* $X \in L^1_G(\Omega)$ *and* $\mathbb{P} \in \mathcal{Q}$, *one has*

$$\mathbb{E}_G[X|\mathcal{F}_t] = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{Q}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[X|\mathcal{F}_t], \ \mathbb{P} - a.s. \tag{2.3}$$

Definition

The quadratic variation of the G-Brownian motion is defined as

$$\langle B \rangle_t = B_t^2 - 2\int_0^t B_s dB_s, \qquad \forall t \leq T,$$

and it is a continuous increasing process which is absolutely continuous with respect to dt.

- Here $\langle B \rangle$ perfectly characterizes the part of uncertainty, or ambiguity, of B.
- For $s, t \geq 0$, we have that $\langle B \rangle_{s+t} - \langle B \rangle_s$ is independent of $\mathcal{F}_s$ and $\langle B \rangle_{s+t} - \langle B \rangle_s \sim \langle B \rangle_t$.
- We say that $\langle B \rangle_t$ is $N([\underline{\sigma}^2 t, \overline{\sigma}^2 t] \times \{0\})$-distributed, i.e., for all $\varphi \in C_{b,Lip}(\mathbb{R})$,

$$\mathbb{E}_G\left[\varphi(\langle B \rangle_t)\right] = \sup_{\underline{\sigma}^2 \leq v \leq \overline{\sigma}^2} \varphi(vt). \tag{2.4}$$

G-Martingales

Definition

A process $M = (M_t)_{t\in[0,T]}$ is called G-martingale if $\mathbb{E}_G[|M_t|] < \infty$ for all $t \in [0,T]$ and $E_G[M_t|\mathcal{F}_s] = M_s$ for all $s \le t \le T$. If M and $-M$ are both G-martingales, M is called a symmetric G-martingale.

Theorem (Theorem 4.5 of Song [6])

Let $\beta > 1$ and $H \in L_G^\beta(\mathcal{F}_T)$. Then the G-martingale M with $M_t := \mathbb{E}_G[H|\mathcal{F}_t]$, $t \in [0,T]$, has the following representation

$$M_t = X_0 + \int_0^t \theta_s dB_s - K_t,$$

where K is a continuous, increasing process with $K_0 = 0$, $K_T \in L_G^\alpha(\mathcal{F}_T)$, $(\theta_t)_{t\in[0,T]} \in M_G^\alpha(0,T)$, $\forall \alpha \in [1,\beta)$, and $-K$ is a G-martingale.

Theorem (Theorem 2.2 of Peng [4])

Let $H \in L_{ip}(\Omega_T)$, then for every $0 \leq t \leq T$ we have

$$\mathbb{E}_G[H|\mathcal{F}_t] = \mathbb{E}_G[H] + \int_0^t \theta_s dB_s + \int_0^t \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds, \quad (2.5)$$

where $(\theta_t)_{t\in[0,T]} \in M_G^2(0,T)$ and $(\eta_t)_{t\in[0,T]} \in M_G^1(0,T)$.

The Setting

- Consider a finite time horizon T.
- The financial market consists of two primary assets:

$$\begin{cases} dX_t = X_t dB_t, & X_0 > 0, \\ d\gamma_t = 0, & \gamma_0 = 1, \end{cases}$$

where B is a G-Brownian motion.

Definition

A trading strategy $\varphi = (\phi_t, \xi_t)_{t\in[0,T]}$ *with value* $V_t(\varphi) = \phi_t X_t + \xi_t$ *is called admissible self-financing if* $(\phi_t)_{t\in[0,T]} \in \Phi$, *where*

$$\Phi := \left\{ \phi \text{ predictable} \middle| \ \mathbb{E}_G \left[\left(\int_0^T \phi_t X_t dB_t \right)^2 \right] < \infty \right\},$$

and $V_t(\varphi) = V_0 + \int_0^t \phi_s dX_s$, $\forall \ t \in [0, T]$.

- **Mean-variance hedging**: Given $H \in L_G^{2+\epsilon}(\mathcal{F}_T)$, for an $\epsilon > 0$, minimize the residual terminal risk defined as

$$J_0(V_0,\phi) := \mathbb{E}_G\left[(H - V_T(V_0,\phi))^2\right] = \sup_{P\in\mathcal{P}} E^P\left[(H - V_T(V_0,\phi))^2\right] \tag{3.1}$$

by the choice of $(V_0,\phi) \in \mathbb{R}_+ \times \Phi$.

Lemma

The initial wealth of the optimal mean-variance portfolio V_0^* lies in the interval $[-\mathbb{E}_G[-H], \mathbb{E}_G[H]]$.

Proposition

There exists a unique solution for the mean-variance hedging problem, i.e.

$$\inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} \mathbb{E}_G\left[(H - V_T(V_0,\phi))^2\right] = \mathbb{E}_G\left[(H - V_T(V_0^*,\phi^*))^2\right], \quad (3.2)$$

for $(V_0^*,\phi^*) \in \mathbb{R}_+ \times \Phi$.

Mean-Variance Hedging

- We consider $H \in L_G^{2+\epsilon}(\mathcal{F}_T)$ with decomposition

$$
\begin{aligned}
H &= \mathbb{E}_G[H] + \int_0^T \theta_s dB_s - K_T(\eta) \\
&= \mathbb{E}_G[H] + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s) ds.
\end{aligned} \tag{4.1}
$$

- Any random variable in $L_G^{2+\epsilon}(\mathcal{F}_T)$ is the limit in the $L_G^{2+\epsilon}$-norm of elements in $L_{ip}(\Omega_T)$.

Theorem

Let be given a claim $H \in L_G^{2+\epsilon}(\mathcal{F}_T)$ *and a sequence of random variables* $(H^n)_{n\in\mathbb{N}}$ *such that* $\|H - H^n\|_{2+\epsilon} \to 0$ *as* $n \to \infty$*. Then as* $n \to \infty$ *we have*

$$J_n^* \to J^*,$$

where, for every $n \in \mathbb{N}$,

$$J_n^* := \inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} \mathbb{E}_G\left[\left(H^n - V_T(V_0,\phi)\right)^2\right]$$

and

$$J^* := \inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} \mathbb{E}_G\left[\left(H - V_T(V_0,\phi)\right)^2\right].$$

Proposition

Consider a claim $H \in L_G^{2+\epsilon}(\mathcal{F}_T)$ *of the following form*

$$H = \mathbb{E}_G[H] + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B \rangle_s - \int_0^T 2G(\eta_s) ds, \tag{4.2}$$

where $\theta \in M_G^2(0, T)$ *and* $\eta \in M_G^1(0, T)$ *is a deterministic process. The optimal mean-variance portfolio is given by*

$$(V_0^*, \phi^*) = (\frac{\mathbb{E}_G[H] - \mathbb{E}_G[-H]}{2}, \frac{\theta}{X}).$$

- **Example I**: For $H = c + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds$ in $L_{ip}(\Omega_T)$, then η is deterministic if and only if

$$H = \frac{a(T)}{2} B_T^2 + b(T)B_T + c(T),$$

for a, b, c deterministic functions.

- **Example II**: If $H = \Phi(e^{B_T})$ for some real valued Lipschitz function Φ, then η is deterministic if and only if

$$H = a(T)B_T + b(T)e^{B_T} + c(T),$$

for a, b, c deterministic functions, see also Vorbrink [7].

Theorem (Mean Uncertainty)

Let $H \in L_G^{2+\epsilon}(\mathcal{F}_T)$ *be of the form*

$$H = \mathbb{E}_G[H] + \int_0^T \theta_s dB_s + \int_0^T \psi(\langle B\rangle_s) d\langle B\rangle_s - 2\int_0^T G(\psi(\langle B\rangle_s))ds,$$

where $(\theta_t)_{t\in[0,T]} \in M_G^2(0,T)$ *and* $\psi : \mathbb{R} \to \mathbb{R}$ *is such that there exists* $k \in \mathbb{R}$ *for which*

$$|\psi(x) - \psi(y)| \leq |x - y|^k,$$

for all $x, y \in \mathbb{R}$. *The optimal mean-variance portfolio is given by*

$$(V_0^*, \phi^*) = \left(\frac{\mathbb{E}_G[H] - \mathbb{E}_G[-H]}{2}, \frac{\theta}{X}\right).$$

- We can characterise the class of contingent claims with η given by a function with polynomial growth of $\langle B \rangle$. This set includes the family of Lipschitz function of $\langle B \rangle$.

- This includes **volatility swaps**, i.e. $H = \sqrt{\langle B \rangle_T} - K$ with $K \in \mathbb{R}_+$, and other volatility derivatives.

- In fact, given a Lipschitz function Φ, the claim $\Phi(\langle B\rangle_T)$ can be written as

$$\Phi(\langle B\rangle_T) = \mathbb{E}_G\left[\Phi(\langle B\rangle_T)\right] + \int_0^T \partial_x u(s,\langle B\rangle_s)\langle B\rangle_s d\langle B\rangle_s - 2\int_0^T G(\partial_x u(s,\langle B\rangle_s))\langle B\rangle_s ds,$$

where $u(t,x)$ solves

$$\begin{cases} \partial_t u + 2G(x\partial_x u) = 0, \\ u(T,x) = \Phi(x), \end{cases}$$

as a consequence of the nonlinear Feynman-Kac formula for G-Brownian motion (see Peng [4]) and the G-Itô formula (see Peng [2]).

- Denote

$$H = \mathbb{E}_G[H] + \theta_{t_1}\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2, \qquad (4.3)$$

where $0 \leq t_1 < t_2 \leq T$, $\theta_{t_1} \in L^2_G(\mathcal{F}_{t_1})$, $\Delta B_{t_2} := B_{t_2} - B_{t_1}$ and similarly for $\Delta\langle B\rangle_{t_2}$ and Δt_2.

Theorem (Mean and volatility uncertainty)

Consider a claim $H \in L^{2+\epsilon}_G(\mathcal{F}_T)$ with decomposition as in (4.3). The optimal mean-variance portfolio is given by (V_0^*, ϕ^*), where $\phi^* = \theta/X$ and V_0^* solves

$$\inf_{V_0} \mathbb{E}_G\Big[(\mathbb{E}_G[H] - V_0)^2 \vee (\mathbb{E}_G[H] - V_0 - (\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2|\eta_{t_1}|)^2\Big]. \qquad (4.4)$$

Example

Let H be

$$H = \mathbb{E}_G[H] + \theta_{t_i} \Delta B_{t_{i+1}} + \eta_{t_i} \Delta \langle B \rangle_{t_{i+1}} - 2G(\eta_{t_i}) \Delta t_{i+1},$$

where $\theta_{t_i} \in L^2_G(\mathcal{F}_{t_i})$ *and* $\eta_{t_i} = e^{B_{t_i}}$. *The optimal initial wealth of the mean-variance portfolio is different from* $\frac{\mathbb{E}_G[H] - \mathbb{E}_G[-H]}{2}$.

Consider a claim $H \in L_G^{2+\epsilon}(\mathcal{F}_T)$ of the form

$$H = \mathbb{E}_G[H] + \int_0^{t_2} \theta_s dB_s + \eta_{t_0} \Delta \langle B \rangle_{t_1} - 2G(\eta_{t_0}) \Delta t_1 + \eta_{t_1} \Delta \langle B \rangle_{t_2} - 2G(\eta_{t_1}) \Delta t_2, \tag{4.5}$$

where $0 = t_0 < t_1 < t_2 = T$, $(\theta_s)_{s \in [0,t_2]} \in M_G^2(0, t_2)$, $\eta_{t_0} \in \mathbb{R}$, $\eta_{t_1} \in L_G^2(\mathcal{F}_{t_1})$ and

$$|\eta_{t_1}| = \mathbb{E}_G\left[|\eta_{t_1}|\right] + \int_0^{t_1} \mu_s dB_s + \xi_{t_0} \Delta \langle B \rangle_{t_1} - 2G(\xi_{t_0}) \Delta t_1, \tag{4.6}$$

for $(\mu_s)_{s \in [0,t_1]} \in M_G^2(0, t_1)$ and $\xi_{t_0} \in \mathbb{R}$.

Theorem

The optimal mean-variance portfolio for $H \in L_G^{2+\epsilon}(\mathcal{F}_T)$ of the form (4.5) is given by

$$\phi_t^* X_t = \left(\theta_t - \frac{\mu_t(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2}{2}\right)\mathbb{I}_{(t_0,t_1]}(t) + \theta_t \mathbb{I}_{(t_1,t_2]}(t)$$

and

$$V_0^* = \mathbb{E}_G[H] - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 \mathbb{E}_G\left[|\eta_{t_1}|\right] - \epsilon,$$

where $\epsilon \in \mathbb{R}$ solves

$$\inf_{\epsilon} \mathbb{E}_G\Bigg[\Bigg(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 + \Bigg|\epsilon + \left(\eta_{t_0} - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\xi_{t_0}\Delta t_1\right)\Delta\langle B\rangle_{t_1} + \\ - 2\left(G(\eta_{t_0}) - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_1 G(\xi_{t_0})\right)\Delta t_1\Bigg|\Bigg)^2\Bigg].$$

Thank you for your attention!

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