

Robust Quadratic Hedging via G-Expectation

Francesca Biagini

Mathematisches Institut Ludwig-Maximilians-Universität München

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Introduction

- We consider the problem of mean-variance hedging in the context of volatility uncertainty, within the *G*-expectation framework.
- This talk is based on Biagini, F., Mancin, J., Meyer-Brandis, T., *Robust Mean-Variance Hedging via G-Expectation*, Preprint LMU, 2016.

Preliminaries (Peng [2])

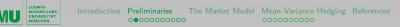
Let Ω be a given set and \mathcal{H} be a vector lattice of real functions defined on Ω , i.e. a linear space containing 1 such that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$.

Definition

A sublinear expectation $\mathbb E$ is a functional $\mathbb E\colon \mathcal H\to\mathbb R$ satisfying the following properties

- 1. Monotonicity: If $X, Y \in \mathcal{H}$ and $X \ge Y$ then $\mathbb{E}[X] \ge \mathbb{E}[Y]$.
- 2. Constant preserving: For all $c \in \mathbb{R}$ we have $\mathbb{E}[c] = c$.
- 3. Sub-additivity: For all $X, Y \in \mathcal{H}$ we have $\mathbb{E}[X] \mathbb{E}[Y] \leq \mathbb{E}[X Y]$.
- 4. **Positive homogeneity:** For all $X \in \mathcal{H}$ we have $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \ge 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.



We consider a space \mathcal{H} of random variables having the following property: if $X_i \in \mathcal{H}, i = 1, ..., n$ then

$$\phi(X_1,\ldots,X_n)\in\mathfrak{H},\quad\forall\phi\in C_{b,Lip}(\mathbb{R}^n),$$

where $C_{b,Lip}(\mathbb{R}^n)$ is the space of all bounded Lipschitz continuous functions on \mathbb{R}^n .

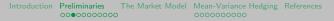
Definition

An *m*-dimensional random vector $Y = (Y_1, \ldots, Y_m)$ is said to be independent of an *n*-dimensional random vector $X = (X_1, \ldots, X_n)$ if for every $\phi \in C_{b,Lip}(\mathbb{R}^n \times \mathbb{R}^m)$

$$\mathbb{E}[\phi(X,Y)] = \mathbb{E}[\mathbb{E}[\phi(x,Y)]_{x=X}].$$

If n = m, we say that X and Y are identically distributed $(X \sim Y)$, if for each $\phi \in C_{b,Lip}(\mathbb{R}^n)$

$$\mathbb{E}[\phi(X)] = \mathbb{E}[\phi(Y)].$$



Definition

A random variable X on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G-normal distributed if for any $a, b \ge 0$

$$aX + bar{X} \sim \sqrt{a^2 + b^2}X,$$

where \overline{X} is an independent copy of X. Such X is symmetric, i.e. $\mathbb{E}(X) = \mathbb{E}(-X) = 0$.

The letter G denotes the function

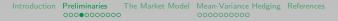
$$G(y) := rac{1}{2}\mathbb{E}(yX^2): \mathbb{R} \mapsto \mathbb{R}.$$

We have the following identity

$$G(y) = \frac{1}{2}\overline{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-,$$
 with $\overline{\sigma}^2 := \mathbb{E}(X^2)$ and $\underline{\sigma}^2 := -\mathbb{E}(-X^2).$

Francesca Biagini

University of Southern California

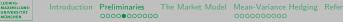


Definition

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A stochastic process $B = (B_t)_{t \ge 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a *G*-Brownian motion if it satisfies the following conditions

- 1. $B_0 = 0$,
- 2. $B_t \in \mathcal{H}$ for each $t \geq 0$.
- 3. For each $t, s \ge 0$ the increment $B_{t+s} B_t$ is independent of $(B_{t_1}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \le t_1 < \ldots < t_n \le t$.
- 4. $(B_{t+s} B_t)s^{-1/2}$ is *G*-normally distributed.
 - It is possible to choose a sub-linear space such that the canonical process is a *G*-Brownian motion. In this case the corresponding sub-linear expectation 𝔼_G is called *G*-expectation.



• Following Peng [3] and Denis, Hu, and Peng [1], we introduce the following notation: for each $t \in [0, \infty)$

1.
$$\Omega = C_0(\mathbb{R}_+, \mathbb{R}), \ \Omega_t := \{\omega_{.\wedge t} : \omega \in \Omega\}, \ \mathcal{F}_t := \mathcal{B}(\Omega_t)$$

- 2. $L_{ip}(\Omega_t) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) | n \in \mathbb{N}, t_1, \ldots, t_n \in [0, t], \varphi \in C_{b, Lip}(\mathbb{R}^n)\}$
- 3. For $p \ge 1$, $L_G^p(\mathcal{F}_T)$ is the completion of $L_{ip}(\Omega_T)$ under the norm $\|\xi\|_p := \mathbb{E}_G[|\xi|^p]^{\frac{1}{p}}$.
- for p ≥ 1, M^p_G(0, T) is the completion of the set of elementary processes of the form

$$\eta(t) = \sum_{i=1}^{n-1} \xi_i I_{[t_i, t_{i+1})}(s),$$

where $0 \le t_1 < t_2 < \ldots < t_n \le T$, $n \ge 1$ and $\xi_i \in Lip(\Omega_{t_i})$ under the norm

$$\|\eta\|_{M^p_G(0,T)} := \mathbb{E}_G[\int_0^T |\eta(s)|^p ds]^{1/p}.$$



Definition

Consider

$$X = \phi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}), \quad \phi \in C_{b, Lip}(\mathbb{R}^{d \times n})$$

for $0 \le t_1 < \ldots < t_n < \infty$. We define the conditional *G*-expectation under \mathcal{F}_{t_1} as

$$\mathbb{E}_{G}[X|\mathcal{F}_{t_{1}}] := \psi(B_{t_{1}}, B_{t_{2}} - B_{t_{1}}, \dots, B_{t_{j}} - B_{t_{j-1}}),$$

where

$$\psi(x) := \mathbb{E}_{G}[\phi(x, B_{t_{j+1}} - B_{t_{j}}, \dots, B_{t_{n}} - B_{t_{n-1}})].$$



Theorem (Denis, Hu, and Peng [1], Theorem 52 and 54) Let $(\tilde{\Omega}, \mathcal{G}, \mathbb{P}_0)$ be a probability space carrying a standard Brownian motion W with respect to its natural filtration \mathbb{G} . Let $\Theta = [\underline{\sigma}, \overline{\sigma}]$ and denote by $\mathcal{A}_{0,\infty}^{\Theta}$ the set of all Θ -valued \mathbb{G} -adapted processes on $[0, \infty)$. For each $\theta \in \mathcal{A}_{0,\infty}^{\Theta}$ define \mathbb{P}^{θ} as the law of a stochastic integral $\int_0^{\cdot} \theta_s dW_s$ on the canonical space $\Omega = C_0(\mathbb{R}_+, \mathbb{R})$. We introduce the sets

$$\mathcal{P}_1 := \{ \mathbb{P}^{\theta} \colon \theta \in \mathcal{A}_{0,\infty}^{\Theta} \}, \quad and \quad \mathcal{P} := \overline{\mathcal{P}_1}, \tag{2.1}$$

where the closure is taken in the weak topology. Then we have the representation

$$\mathbb{E}_{G}[X] = \sup_{\mathbb{P} \in \mathcal{P}_{1}} \mathbb{E}^{\mathbb{P}}[X] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[X], \quad for \ each \ X \in L^{1}_{G}(\Omega).$$
(2.2)

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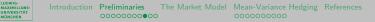


• Similarly an analogous representation holds for the *G*-conditional expectation.

Proposition (Soner, Touzi, and Zhang [5], Proposition 3.4)

Let $\Omega(t, \mathbb{P}) := \{\mathbb{P}' \in \Omega : \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_t\}$, where $\Omega = \mathcal{P}$ or \mathcal{P}_1 . Then for any $X \in L^1_G(\Omega)$ and $\mathbb{P} \in \Omega$, one has

$$\mathbb{E}_{G}[X|\mathcal{F}_{t}] = \underset{\mathbb{P}' \in \mathfrak{Q}(t,\mathbb{P})}{\operatorname{ess \, sup}} \mathbb{E}^{\mathbb{P}'}[X|\mathcal{F}_{t}], \ \mathbb{P}-a.s.$$
(2.3)



Definition

The quadratic variation of the G-Brownian motion is defined as

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s dB_s, \qquad \forall t \leq T,$$

and it is a continuous increasing process which is absolutely continuous with respect to dt.

- Here $\langle B \rangle$ perfectly characterizes the part of uncertainty, or ambiguity, of B.
- For $s, t \ge 0$, we have that $\langle B \rangle_{s+t} \langle B \rangle_s$ is independent of \mathcal{F}_s and $\langle B \rangle_{s+t} \langle B \rangle_s \sim \langle B \rangle_t$.
- We say that $\langle B \rangle_t$ is $N([\underline{\sigma}^2 t, \overline{\sigma}^2 t] \times \{0\})$ -distributed, i.e., for all $\varphi \in C_{b,Lip}(\mathbb{R})$,

$$\mathbb{E}_{G}\left[\varphi(\langle B\rangle_{t})\right] = \sup_{\underline{\sigma}^{2} \le v \le \overline{\sigma}^{2}} \varphi(vt).$$
(2.4)



G-Martingales

Definition

A process $M = (M_t)_{t \in [0,T]}$ is called G-martingale if $\mathbb{E}_G[|M_t|] < \infty$ for all $t \in [0,T]$ and $E_G[M_t|\mathcal{F}_s] = M_s$ for all $s \leq t \leq T$. If M and -M are both G-martingales, M is called a symmetric G-martingale.

Theorem (Theorem 4.5 of Song [6]) Let $\beta > 1$ and $H \in L_G^{\beta}(\mathcal{F}_T)$. Then the G-martingale M with $M_t := \mathbb{E}_G[H|\mathcal{F}_t]$, $t \in [0, T]$, has the following representation

$$M_t = X_0 + \int_0^t \theta_s dB_s - K_t,$$

where K is a continuous, increasing process with $K_0 = 0$, $K_T \in L^{\alpha}_G(\mathcal{F}_T)$, $(\theta_t)_{t \in [0,T]} \in M^{\alpha}_G(0,T)$, $\forall \alpha \in [1,\beta)$, and -K is a G-martingale.



Theorem (Theorem 2.2 of Peng [4]) Let $H \in L_{ip}(\Omega_T)$, then for every $0 \le t \le T$ we have

$$\mathbb{E}_{G}[H|\mathcal{F}_{t}] = \mathbb{E}_{G}[H] + \int_{0}^{t} \theta_{s} dB_{s} + \int_{0}^{t} \eta_{s} d\langle B \rangle_{s} - 2 \int_{0}^{T} G(\eta_{s}) ds, \quad (2.5)$$

where $(\theta_{t})_{t \in [0,T]} \in M_{G}^{2}(0,T)$ and $(\eta_{t})_{t \in [0,T]} \in M_{G}^{1}(0,T).$

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The Setting

- Consider a finite time horizon T.
- The financial market consists of two primary assets:

$$\begin{cases} dX_t = X_t dB_t, & X_0 > 0, \\ d\gamma_t = 0, & \gamma_0 = 1, \end{cases}$$

where B is a G-Brownian motion.

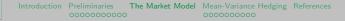


Definition

A trading strategy $\varphi = (\phi_t, \xi_t)_{t \in [0,T]}$ with value $V_t(\varphi) = \phi_t X_t + \xi_t$ is called admissible self-financing if $(\phi_t)_{t \in [0,T]} \in \Phi$, where

$$\Phi := \left\{ \phi \text{ predictable} | \mathbb{E}_{G} \left[\left(\int_{0}^{T} \phi_{t} X_{t} dB_{t} \right)^{2} \right] < \infty \right\},$$

and $V_t(\varphi) = V_0 + \int_0^t \phi_s dX_s$, $\forall t \in [0, T]$.



 Mean-variance hedging: Given H ∈ L^{2+ε}_G(𝔅_T), for an ε > 0, minimize the residual terminal risk defined as

$$J_{0}(V_{0},\phi) := \mathbb{E}_{G}\left[(H - V_{T}(V_{0},\phi))^{2} \right] = \sup_{P \in \mathcal{P}} E^{P} \left[(H - V_{T}(V_{0},\phi))^{2} \right]$$
(3.1)

by the choice of $(V_0, \phi) \in \mathbb{R}_+ \times \Phi$.

Lemma

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The initial wealth of the optimal mean-variance portfolio V_0^* lies in the interval $[-\mathbb{E}_G[-H], \mathbb{E}_G[H]]$.



Proposition

There exists a unique solution for the mean-variance hedging problem, i.e.

$$\inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi}\mathbb{E}_G\left[(H-V_T(V_0,\phi))^2\right] = \mathbb{E}_G\left[(H-V_T(V_0^*,\phi^*))^2\right], \quad (3.2)$$

For $(V_0^*,\phi^*)\in\mathbb{R}_+\times\Phi.$



Mean-Variance Hedging

• We consider $H \in L^{2+\epsilon}_{G}(\mathcal{F}_{\mathcal{T}})$ with decomposition

$$H = \mathbb{E}_{G} [H] + \int_{0}^{T} \theta_{s} dB_{s} - K_{T}(\eta)$$

= $\mathbb{E}_{G} [H] + \int_{0}^{T} \theta_{s} dB_{s} + \int_{0}^{T} \eta_{s} d\langle B \rangle_{s} - 2 \int_{0}^{T} G(\eta_{s}) ds.$ (4.1)

• Any random variable in $L^{2+\epsilon}_{G}(\mathcal{F}_{T})$ is the limit in the $L^{2+\epsilon}_{G}$ -norm of elements in $L_{ip}(\Omega_{T})$.



Theorem

Let be given a claim $H \in L^{2+\epsilon}_G(\mathcal{F}_T)$ and a sequence of random variables $(H^n)_{n\in\mathbb{N}}$ such that $\|H - H^n\|_{2+\epsilon} \to 0$ as $n \to \infty$. Then as $n \to \infty$ we have

$$J_n^* \to J^*,$$

where, for every $n \in \mathbb{N}$,

$$J_n^* := \inf_{(V_0,\phi) \in \mathbb{R}_+ imes \Phi} \mathbb{E}_G \left[\left(H^n - V_T(V_0,\phi) \right)^2 \right]$$

and

$$J^* := \inf_{(V_0,\phi) \in \mathbb{R}_+ imes \Phi} \mathbb{E}_G \left[(H - V_T(V_0,\phi))^2 \right].$$



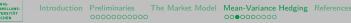
Proposition

Consider a claim $H \in L^{2+\epsilon}_{G}(\mathfrak{F}_{T})$ of the following form

$$H = \mathbb{E}_{G}[H] + \int_{0}^{T} \theta_{s} dB_{s} + \int_{0}^{T} \eta_{s} d\langle B \rangle_{s} - \int_{0}^{T} 2G(\eta_{s}) ds, \qquad (4.2)$$

where $\theta \in M^2_G(0, T)$ and $\eta \in M^1_G(0, T)$ is a deterministic process. The optimal mean-variance portfolio is given by

$$(V_0^*, \phi^*) = (\frac{\mathbb{E}_G[H] - \mathbb{E}_G[-H]}{2}, \frac{\theta}{X}).$$



• Example I: For $H = c + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds$ in $L_{ip}(\Omega_T)$, then η is deterministic if and only if

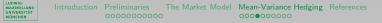
$$H=\frac{a(T)}{2}B_T^2+b(T)B_T+c(T),$$

for a, b, c deterministic functions.

 Example II: If H = Φ(e^{B_T}) for some real valued Lipschitz function Φ, then η is deterministic if and only if

$$H = a(T)B_T + b(T)e^{B_T} + c(T),$$

for a, b, c deterministic functions, see also Vorbrink [7].



Theorem (**Mean Uncertainty**) Let $H \in L^{2+\epsilon}_G(\mathfrak{F}_T)$ be of the form

$$H = \mathbb{E}_{G}[H] + \int_{0}^{T} \theta_{s} dB_{s} + \int_{0}^{T} \psi(\langle B \rangle_{s}) d\langle B \rangle_{s} - 2 \int_{0}^{T} G(\psi(\langle B \rangle_{s})) ds,$$

where $(\theta_t)_{t \in [0,T]} \in M^2_G(0,T)$ and $\psi : \mathbb{R} \to \mathbb{R}$ is such that there exists $k \in \mathbb{R}$ for which

$$|\psi(x)-\psi(y)|\leq |x-y|^k,$$

for all $x, y \in \mathbb{R}$. The optimal mean-variance portfolio is given by

$$(V_0^*,\phi^*)=(\frac{\mathbb{E}_G[H]-\mathbb{E}_G[-H]}{2},\frac{\theta}{X}).$$



- We can characterise the class of contingent claims with η given by a function with polynomial growth of (B). This set includes the family of Lipschitz function of (B).
- This includes volatility swaps, i.e. $H = \sqrt{\langle B \rangle_T} K$ with $K \in \mathbb{R}_+$, and other volatility derivatives.

 In fact, given a Lipschitz function Φ, the claim Φ(⟨B⟩_T) can be written as

$$\Phi(\langle B \rangle_T) = \mathbb{E}_G \left[\Phi(\langle B \rangle_T) \right] + \int_0^T \partial_x u(s, \langle B \rangle_s) \langle B \rangle_s d\langle B \rangle_s - 2 \int_0^T G(\partial_x u(s, \langle B \rangle_s)) \langle B \rangle_s ds,$$

where u(t, x) solves

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$$\begin{cases} \partial_t u + 2G(x\partial_x u) = 0, \\ u(T, x) = \Phi(x), \end{cases}$$

as a consequence of the nonlinear Feynman-Kac formula for *G*-Brownian motion (see Peng [4]) and the *G*-Itô formula (see Peng [2]).



Denote

$$H = \mathbb{E}_{G}[H] + \theta_{t_1} \Delta B_{t_2} + \eta_{t_1} \Delta \langle B \rangle_{t_2} - 2G(\eta_{t_1}) \Delta t_2, \qquad (4.3)$$

where $0 \leq t_1 < t_2 \leq T$, $\theta_{t_1} \in L^2_G(\mathcal{F}_{t_1})$, $\Delta B_{t_2} := B_{t_2} - B_{t_1}$ and similarly for $\Delta \langle B \rangle_{t_2}$ and Δt_2 .

Theorem (Mean and volatility uncertainty)

Consider a claim $H \in L^{2+\epsilon}_{G}(\mathcal{F}_{T})$ with decomposition as in (4.3). The optimal mean-variance portfolio is given by (V_{0}^{*}, ϕ^{*}) , where $\phi^{*} = \theta/X$ and V_{0}^{*} solves

$$\inf_{V_0} \mathbb{E}_G \Big[(\mathbb{E}_G[H] - V_0)^2 \lor (\mathbb{E}_G[H] - V_0 - (\overline{\sigma}^2 - \underline{\sigma}^2) \Delta t_2 |\eta_{t_1}|)^2 \Big].$$
(4.4)



Example

Let H be

$$H = \mathbb{E}_{G}[H] + \theta_{t_{i}} \Delta B_{t_{i+1}} + \eta_{t_{i}} \Delta \langle B \rangle_{t_{i+1}} - 2G(\eta_{t_{i}}) \Delta t_{i+1},$$

where $\theta_{t_i} \in L^2_G(\mathcal{F}_{t_i})$ and $\eta_{t_i} = e^{B_{t_i}}$. The optimal initial wealth of the mean-variance portfolio is different from $\frac{\mathbb{E}_G[H] - \mathbb{E}_G[-H]}{2}$.



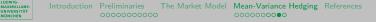
Consider a claim $H \in L^{2+\epsilon}_{G}(\mathcal{F}_{T})$ of the form

$$H = \mathbb{E}_{G}[H] + \int_{0}^{t_{2}} \theta_{s} dB_{s} + \eta_{t_{0}} \Delta \langle B \rangle_{t_{1}} - 2G(\eta_{t_{0}}) \Delta t_{1} + \eta_{t_{1}} \Delta \langle B \rangle_{t_{2}} - 2G(\eta_{t_{1}}) \Delta t_{2},$$
(4.5)

where $0 = t_0 < t_1 < t_2 = T$, $(\theta_s)_{s \in [0, t_2]} \in M^2_G(0, t_2)$, $\eta_{t_0} \in \mathbb{R}$, $\eta_{t_1} \in L^2_G(\mathcal{F}_{t_1})$ and

$$|\eta_{t_1}| = \mathbb{E}_G[|\eta_{t_1}|] + \int_0^{t_1} \mu_s dB_s + \xi_{t_0} \Delta \langle B \rangle_{t_1} - 2G(\xi_{t_0}) \Delta t_1, \qquad (4.6)$$

for $(\mu_s)_{s\in[0,t_1]}\in M^2_G(0,t_1)$ and $\xi_{t_0}\in\mathbb{R}$.



Theorem

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The optimal mean-variance portfolio for $H \in L^{2+\epsilon}_G(\mathcal{F}_T)$ of the form (4.5) is given by

$$\phi_t^* X_t = \left(heta_t - rac{\mu_t(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2}{2}
ight) \mathbb{I}_{(t_0,t_1]}(t) + heta_t \mathbb{I}_{(t_1,t_2]}(t)$$

and

$$V_0^* = \mathbb{E}_G[H] - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 \mathbb{E}_G[|\eta_{t_1}|] - \epsilon,$$

where $\epsilon \in \mathbb{R}$ solves

$$\begin{split} \inf_{\epsilon} \mathbb{E}_{G} \bigg[\bigg(\frac{|\eta_{t_{1}}|}{2} (\overline{\sigma}^{2} - \underline{\sigma}^{2}) \Delta t_{2} + \Big| \epsilon + \bigg(\eta_{t_{0}} - \frac{1}{2} (\overline{\sigma}^{2} - \underline{\sigma}^{2}) \xi_{t_{0}} \Delta t_{1} \bigg) \Delta \langle B \rangle_{t_{1}} + \\ &- 2 \left(G(\eta_{t_{0}}) - \frac{1}{2} (\overline{\sigma}^{2} - \underline{\sigma}^{2}) \Delta t_{1} G(\xi_{t_{0}}) \right) \Delta t_{1} \bigg| \bigg)^{2} \bigg]. \end{split}$$



Thank you for your attention!



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