Causal optimal transport and its links to enlargement of filtrations and stochastic optimization problems

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joint work with J. Backhoff Veraguas and A. Zalashko

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Main questions

Given B, Brownian motion in its own filtration $\mathcal{F}^B$, and given a bigger filtration $\mathcal{F}^{B,G} \supseteq \mathcal{F}^B$:

- When is B going to remain a semimartingale? ($B_t = \tilde{B}_t + A_t$)
  In particular, when will it have an absolutely continuous finite variation part? $\left(B_t = \tilde{B}_t + \int_0^t a_s ds\right)$
  [same questions for any continuous semimartingale]

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  [same questions for any continuous semimartingale]
- How can we estimate the value of the additional information ($\mathcal{F}^{B,G}$ vs $\mathcal{F}^B$) in terms of stochastic optimization problems (optimal value w.r.to $\mathcal{F}^{B,G}$ vs optimal value w.r.to $\mathcal{F}^B$)?

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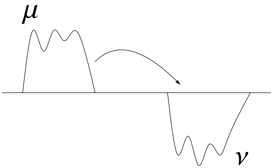
$\longrightarrow$ We will answer the above questions via causal transport.

Outline

Brushing up on classical mass transport

Monge-Kantorovich problem. Given two Polish probability spaces $(\mathcal{X},\mu),(\mathcal{Y},\nu)$, "move the mass" from μ to ν minimizing the cost of transportation $c:\mathcal{X}\times\mathcal{Y}\to[0,\infty]$



$$P := \inf\{\mathbb{E}^{\pi}[c(x,y)] : \pi\in\Pi(\mu,\nu)\},$$

$\Pi(\mu,\nu)$: probability measures on $\mathcal{X}\times\mathcal{Y}$ with marginals μ and ν.

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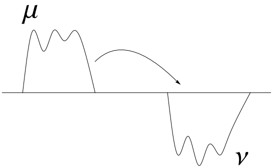

$$P := \inf \{ \mathbb{E}^{\pi}[c(x, y)] : \pi \in \Pi(\mu, \nu) \},$$

$\Pi(\mu, \nu)$: probability measures on $\mathcal{X} \times \mathcal{Y}$ with marginals μ and ν.

$\to$ For c lower semi-continuous, duality $P = D$ holds, where

$$D := \sup \{ \mathbb{E}^{\mu}[\varphi] + \mathbb{E}^{\nu}[\psi] : \varphi \in C_b(\mathcal{X}), \psi \in C_b(\mathcal{Y}), \varphi(x) + \psi(y) \le c(x, y) \}$$

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- We will study constrained primal/dual problems and use them to tackle the above questions.

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$\rightarrow$ In a similar spirit: recent application of optimal transport in model-independent finance $\rightarrow$ martingale optimal transport.

What we will do with it

- **Martingale optimal transport**: Suppose traded Call options on S for maturities T_1, T_2, for all strikes $\Rightarrow S_{T_1} \sim \mu$, $S_{T_2} \sim \nu$
  - we "move S_{T_1} to S_{T_2}" $(\mathcal{X} = \mathcal{Y} = \mathbb{R})$
  - along a martingale (martingale constraint)
  - in order to determine robust price of a claim $c(S_{T_1}, S_{T_2})$: $\inf / \sup \{\mathbb{E}^\pi[c(S_{T_1}, S_{T_2})] : \pi \in \Pi(\mu, \nu),\ \pi \text{ is a martingale}\}$

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- **Causal optimal transport**:
  - we will "move process $(X_t)_t$ to $(Y_t)_t$" $(\mathcal{X} = \mathcal{Y} = \text{path space})$
  - under the causality constraint
  - to characterize preservation of the semimartingale property:

$$\inf\{\mathbb{E}^\pi[c(X, Y)] : \pi \in \Pi(\mu, \nu),\ \pi \text{ is causal}\} \qquad c = ?$$

Causal optimal transport

- Polish probability spaces $(\mathcal{X},\mu),(\mathcal{Y},\nu)$, time horizon $T<\infty$
- Right-continuous filtrations $\mathcal{F}^{\mathcal{X}}=(\mathcal{F}_t^{\mathcal{X}})_{t\in[0,T]},\mathcal{F}^{\mathcal{Y}}=(\mathcal{F}_t^{\mathcal{Y}})_{t\in[0,T]}$, with $\mathcal{F}_T^{\mathcal{X}}=\mathcal{B}(\mathcal{X}),\mathcal{F}_T^{\mathcal{Y}}=\mathcal{B}(\mathcal{Y})$

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Definition (Causal transport plans $\Pi^{\mathcal{F}^{\mathcal{X}},\mathcal{F}^{\mathcal{Y}}}(\mu,\nu)$)

A transport plan $\pi\in\Pi(\mu,\nu)$ is called causal between $(\mathcal{X},\mathcal{F}^{\mathcal{X}},\mu)$ and $(\mathcal{Y},\mathcal{F}^{\mathcal{Y}},\nu)$ if, for all $t\in[0,T]$ and $D\in\mathcal{F}_t^{\mathcal{Y}}$, the map

$$\mathcal{X}\ni x\mapsto\pi^x(D)$$

is measurable w.t.to $\mathcal{F}_t^{\mathcal{X}}$ (π^x regular conditional kernel w.r.to $\mathcal{X}$).

Causal optimal transport

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Example (Yamada-Watanabe'71)

Assume weak-existence of the solution to the SDE:

$$dY_t = \sigma(Y_t)dB_t + b(Y_t)dt, \quad b, \sigma \text{ Borel measurable.}$$

Then $(B, Y)_{\#}\mathbb{P}$ is a causal plan between $(\mathcal{C}[0,\infty), \mathcal{F}, B_{\#}\mathbb{P})$ and $(\mathcal{C}[0,\infty), \mathcal{F}, Y_{\#}\mathbb{P})$, where $\mathcal{F}$ is the canonical filtration on $\mathcal{C}[0,\infty)$.

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From a transport point of view: from an observed trajectory of B, the "mass" can be split at each moment of time into Y only based on the information available up to that time. When there is no splitting of mass (Monge transport), a causal plan is then an actual mapping which is further adapted, i.e. strong solution $Y = F(B)$.

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- Here same filtration (as studied in Lassalle (2015)). We will instead consider different filtrations (filtration enlargement).

Causal transport on path space

Our framework:

- $\mathcal{X} = \mathcal{Y} = \mathcal{C} := C_0([0,T])$
- W coordinate process on $\mathcal{C}$: $W_t(\omega) = \omega_t$
- $\mathcal{F}^{\mathcal{X}} = \mathcal{F}$ filtration generated by W: $\mathcal{F}_t := \bigcap_{u>t} \sigma(W_s, s \le u)$
- $\mathcal{F}^{\mathcal{Y}} = \mathcal{G}$ obtained as enlargement of $\mathcal{F}$ with $G = (g_t(W))_t$:

$$\mathcal{G}_t := \bigcap_{\epsilon>0} \mathcal{G}^0_{t+\epsilon}, \quad \mathcal{G}^0_t := \mathcal{F}_t \vee \sigma(\{G_s, s \le t\}).$$

- given two measures μ, ν on $\mathcal{C}$, we will study causal transport plans between $(\mathcal{C}, \mathcal{F}, \mu)$ and $(\mathcal{C}, \mathcal{G}, \nu)$
- we will often consider $\mu = \gamma :=$ Wiener measure on $\mathcal{C}$

Causal coupling

Notations: For a continuous process Z on a $(\Omega, \mathbb{P})$:

$\mathcal{F}^Z := Z^{-1}(\mathcal{F})$ (right-continuous filtration generated by Z on Ω)
$\mathcal{F}^{Z,G} := Z^{-1}(\mathcal{G})$ (enlargement of $\mathcal{F}^Z$ with $G(Z) = (g_t(Z))_{t\in[0,T]}$)

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Ex. Initial enlargement: $g_t(Z) = L \ \forall t \geq 0$, L random var. $\mathcal{F}^Z$-mbl
Ex. Progressive enlargement: $g_t(Z) = \tau \wedge t$, τ random time $\mathcal{F}^Z$-mbl

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Definition (Causal coupling)

A pair (X, Y) of continuous processes on a probability space $(\Omega, \mathbb{P})$, is called a causal coupling w.r.to $\mathcal{F}^X$ and $\mathcal{F}^{Y,G}$ if $(X, Y)_{\#}\mathbb{P}$ is a causal transport plan between $(\mathcal{C}, \mathcal{F}, X_{\#}\mathbb{P})$ and $(\mathcal{C}, \mathcal{G}, Y_{\#}\mathbb{P})$.

Causal coupling

Easy to see, e.g. by Brémaud-Yor (1978):

Remark (Characterizations of causality)

For a pair (X, Y) of continuous processes on $(\Omega, \mathbb{P})$, TFAE:

- *(X, Y) is a causal coupling w.r.to $\mathcal{F}^X$ and $\mathcal{F}^{Y,G}$;*
- *$\mathbb{P}(D_t \mid \mathcal{F}_t^X) = \mathbb{P}(D_t \mid \mathcal{F}_T^X)$ $\mathbb{P}$-a.s., for all $t \in [0, T]$, $D_t \in \mathcal{F}_t^{Y,G}$;*
- *$\mathcal{F}_t^{Y,G}$ cond.indep. $\mathcal{F}_T^X$ given $\mathcal{F}_t^X$ w.r.to $\mathbb{P}$, for all $t \in [0, T]$;*
- *$\mathcal{H}$-hypothesis holds between $\mathcal{F}^X$ and $\mathcal{F}^X \vee \mathcal{F}^{Y,G}$ w.r.to $\mathbb{P}$.*

(every sq.integrable $\mathcal{F}^X$-mart. is a sq.integrable $\mathcal{F}^X \vee \mathcal{F}^{Y,G}$-mart.)

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Mass transport interpretation: At every time the mass transported to the 2^{nd} process is only based on the information on the 1^{st} process up to that time (+ something independent of the whole 1^{st} process).

Causal coupling: Brownian case

Lemma

Let X be a Brownian motion and Y a continuous process on $(\Omega, \mathbb{P})$. Then (X, Y) is a causal coupling w.r.to $\mathcal{F}^X$ and $\mathcal{F}^{Y,G}$ IFF X is a Brownian motion in $\mathcal{F}^X \vee \mathcal{F}^{Y,G}$.

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Motivating example to study causal coupling in a filtration enlargement framework:

Example

Let B be a Brownian motion on $(\Omega, \mathbb{P})$, which remains a semi-martingale w.r.to the enlarged filtration $\mathcal{F}^{B,G}$, with decomposition

$$dB_t = d\tilde{B}_t + dA_t.$$

Then, for any $T > 0$, $(\tilde{B}, B)$ is a causal coupling w.r.to $\mathcal{F}^{\tilde{B}}$ and $\mathcal{F}^{B,G}$, that is, $(\tilde{B}, B)_{\#}\mathbb{P} \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma, \gamma)$.

Optional projections under causality

Notation. Let Λ be $(\mathcal{F} \otimes \mathcal{G})$-adapted, càdlàg, with integrable variation on $[0, T]$, $\Lambda_0 = 0$. We denote by ${}^o\Lambda^{\mathcal{F}}$ (resp. ${}^{\mathcal{F}}\Lambda^o$) its optional (resp. dual optional) projection w.r.t. $(\pi, \mathcal{F} \otimes \{\emptyset, \mathcal{C}\})$.

Proposition

For any $\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\mu, \nu)$: ${}^{\mathcal{F}}\Lambda^o$ and ${}^o\Lambda^{\mathcal{F}}$ are π-indistinguishable.

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Idea: ${}^o\Lambda^{\mathcal{F}}$ formalizes $\mathbb{E}^\pi[\Lambda_t|\mathcal{F}_t]$, ${}^{\mathcal{F}}\Lambda^o$ formalizes $\int_0^t\mathbb{E}^\pi[d\Lambda_s|\mathcal{F}_s]$

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Therefore:

$$\begin{aligned}{}^{\mathcal{F}}\Lambda^o_t &= \int_0^t\mathbb{E}^\pi[d\Lambda_s|\mathcal{F}_s]=\int_0^t\mathbb{E}^\pi[d\Lambda_s|\mathcal{F}_t]=\mathbb{E}^\pi\left[\int_0^t d\Lambda_s\Big|\mathcal{F}_t\right]\\ &= \mathbb{E}^\pi[\Lambda_t|\mathcal{F}_t]={}^o\Lambda^{\mathcal{F}}_t.\end{aligned}$$

Preliminary results

Define the following set of test functions:

$$\mathcal{H} := span\left(\left\{g\left[f - \mathbb{E}^{\mu}[\,f\,|\mathcal{F}_t^{\mathcal{X}}]\right] : f \in C_b(\mathcal{X}), g \in B_b(\mathcal{Y}, \mathcal{F}_t^{\mathcal{Y}}), t \in [0,T]\right\}\right)$$

Lemma

$\pi \in \Pi(\mu, \nu)$ *is causal w.r.to* $\mathcal{F}^{\mathcal{X}}$ *and* $\mathcal{F}^{\mathcal{Y}}$ $\iff$ $\mathbb{E}^{\pi}[\,h\,] = 0 \ \ \forall h \in \mathcal{H}$.

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Lemma

Assume that μ *satisfies the following weak-continuity property:*

$$\forall f \in C_b(\mathcal{X}), t \in [0,T] : \;\; x \mapsto \mathbb{E}^{\mu}[\, f \,|\mathcal{F}_t^{\mathcal{X}}](x) \text{ is continuous.}$$

Then the set $\Pi^{\mathcal{F}^{\mathcal{X}},\mathcal{F}^{\mathcal{Y}}}(\mu,\nu)$ *is compact for weak convergence.*

Remark. Wiener measure γ satisfies weak-continuity.

Duality

Theorem

Let μ be weakly-continuous, and $c : \mathcal{X} \times \mathcal{Y} \to (-\infty, \infty]$ be bounded from below and lsc. Then there is no duality gap:

$$P := \inf_{\pi \in \Pi^{\mathcal{F}^{\mathcal{X}}, \mathcal{F}^{\mathcal{Y}}}(\mu,\nu)} \mathbb{E}^{\pi}[\,c\,] = \sup_{\substack{\phi \in C_b(\mathcal{X}), \psi \in C_b(\mathcal{Y}), h \in \mathcal{H} \\ \phi \oplus \psi \leq c + h}} \left\{ \mathbb{E}^{\mu}[\phi] + \mathbb{E}^{\nu}[\psi] \right\}$$

$$= \sup_{\substack{\psi \in C_b(\mathcal{Y}), h \in \mathcal{H} \\ \psi \leq c + h}} \mathbb{E}^{\nu}[\psi] =: D$$

and the primal problem is attained.

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$\rightarrow$ We will use the causal optimal transport problem P to study semimartingale decompositions in enlarged filtrations.

Semimartingale property

Notations. $(\omega, \bar{\omega})$: generic element in $\mathcal{C} \times \mathcal{C}$, $\gamma =$ Wiener measure, $V_t(Z)$: total variation of a process/path Z up to time t.

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Theorem (I)

Let ν be a measure on $\mathcal{C}$ such that $\nu \ll \gamma$. Then TFAE:

(i) *causal optimal transport* $\inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\nu)} \mathbb{E}^{\pi}[V_T(\bar{\omega} - \omega)] < \infty$,

(ii) *for some continuous $\mathcal{G}$-adapted A s.t.* $\mathbb{E}^{\nu}[V_T(A)] < \infty$,

$$\xi_t(\bar{\omega}) := \bar{\omega}_t - A_t(\bar{\omega}) \text{ is a } (\nu, \mathcal{G})\text{-Brownian motion.}$$

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Moreover, under (i)-(ii): (a) $\hat{\pi} := (\xi, id)_{\#}\nu$ is optimal in (i),
(b) $\forall\, \pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma, \nu)$ with finite cost, the process $\tilde{A}(\omega, \bar{\omega}) := A(\bar{\omega})$ is the $(\pi, \{\emptyset, \mathcal{C}\} \times \mathcal{G})$-dpp of the process $\Lambda_t(\omega, \bar{\omega}) := \bar{\omega}_t - \omega_t$.

Remark. Under $\hat{\pi}$: $\bar{\omega}_t = \underbrace{\omega_t}_{\uparrow\ \mathcal{G}\text{-BM}} + A_t(\bar{\omega})$ (remember the example)

Semimartingale property

Theorem (II)

TFAE:

1. *any process B which is a Brownian motion on some $(\Omega, \mathbb{P})$, remains a semimartingale in the enlarged filtration $\mathcal{F}^{B,G}$;*
2. *the causal transport problem (i) is finite* for some $\nu \sim \gamma$.

Semimartingale property

Theorem (II)

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Remark. When 1.-2. hold: $dB_t = d\widetilde{B}_t + dA_t$, and $(\widetilde{B}, B)$ causal coupling w.r.to $\mathcal{F}^{\widetilde{B}}, \mathcal{F}^{B,G}$. We now consider a variation of the causal problem in (i), to characterize the case where

$$dB_t = d\tilde{B}_t + \alpha_t(B)dt.$$

(e.g., Brownian bridge, initial enlargement with Jacod's condition, progressive enlargement by a random time: Jeulin-Yor's formula)

The absolutely continuous case

Notations. For $h \in C$, define $[h_t]$ as $\dot{h}_t$, when it exists, and $+\infty$ else

Theorem (I')

Let ν be a measure on C such that $\nu \ll \gamma$, and $\rho : \mathbb{R} \to \mathbb{R}_+$ a convex even function s.t. $\rho(+\infty) = +\infty$ and $\rho(0) = 0$. Then TFAE:

(i') *causal optimal transport* $\inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\nu)} \mathbb{E}^\pi \left[\int_0^T \rho([\bar{\omega}_t - \omega_t]) dt \right] < \infty$,

(ii') *for some $\mathcal{G}$-predictable α s.t.* $\mathbb{E}^\nu \left[\int_0^T \rho(\alpha_s) ds \right] < \infty$,

$$\xi_t(\bar{\omega}) := \bar{\omega}_t - \int_0^t \alpha_s(\bar{\omega}) ds \text{ is a } (\nu, \mathcal{G})\text{-Brownian motion.}$$

Moreover, under (i)-(ii): (a) $\hat{\pi} := (\xi, id)_\# \nu$ is optimal in (i),

(b) $\forall\ \pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma, \nu)$ with finite cost, the process $\tilde{\alpha}(\omega, \bar{\omega}) := \alpha(\bar{\omega})$ is the $(\pi, \{\emptyset, C\} \times \mathcal{G})$-pp of the process $[\bar{\omega}_t - \omega_t]$.

The absolutely continuous case

Theorem (II')

TFAE:

1. *any process B which is a Brownian motion on some $(\Omega, \mathbb{P})$, remains a semimartingale in the enlarged filtration $\mathcal{F}^{B,G}$, with absolutely continuous FV part;*
2. *the causal transport problem (i') is finite for some $\nu \sim \gamma$ and some function ρ as in Theorem I' (eqv., for $\rho = |\,.\,|$).*

Cameron-Martin cost

For $\rho(x) = x^2/2$, if causal problem with $\nu = \gamma$ is finite, then:

- $dB_t = d\tilde{B}_t + \alpha_t(B)dt$, with α square integrable
- by Girsanov, B BM w.r.t. $\mathcal{F}^{B,G}$ under new measure $\mathbb{Q}$
- by martingale representation, $\mathcal{H}'$-hypothesis holds for $\mathcal{F}^B$, $\mathcal{F}^{B,G}$, i.e. all $\mathcal{F}^B$-semimartingales are $\mathcal{F}^{B,G}$-semimartingales;

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- in case of initial enlargement with r.v. $L(B)$ with law ℓ, the value of causal problem equals all the following:

$$\frac{1}{2}\mathbb{E}^\gamma[\int_0^T \alpha_t^2 dt] = \mathrm{Ent}(\mathbb{P}|\mathbb{Q}) = \int \mathrm{Ent}(\gamma^{L=x}|\gamma)\ell(dx) = \mathrm{I}(B, L(B)),$$

where $\mathrm{I}(B, L(B)) := \mathrm{Ent}(P_{B,L(B)}|P_B \otimes P_{L(B)})$ is the mutual information between B and $L(B)$.

Extensions

Our results have natural extensions in two directions:

- $\longrightarrow$ **Multidimensional processes.**
- $\longrightarrow$ **General continuous semimartingales:** for non-Brownian processes, generalization of the definition of causality:

$$\mathbb{E}^{\pi}[(\omega_t - \omega_s) f_s(\bar{\omega})] = 0, \qquad 0 \leq s < t \leq T,\ f_s \in L^{\infty}(\mathcal{C}, \mathcal{G}_s, \nu),$$

which leads to analogous results.

In particular, if X continuous semimartingale which remains a semimartingale in the enlarged filtration $\mathcal{F}^{X,G}$, with $X = \widetilde{X} + N$

$\Rightarrow$ the transport plan $(\widetilde{X}, X)_{\#}\mathbb{P}$ satisfies the condition above.

Applications to stochastic optimization

- **Aim:** use causal transport framework to give an estimate of the value of the additional information, for some classical stochastic optimization problems (difference of optimal value of these problems with or without additional information).
- **Idea:** take projection w.r.to causal couplings of the optimizers in the problem with the larger filtration (additional information), so building a feasible element in the problem with the smaller filtration and making a comparison possible.
- Pflug (2009) uses this idea in discrete-time, to gauge the dependence of multistage stochastic programming problems w.r.to different reference probability measures.

Utility maximisation

- B d-dimensional Brownian motion on $(\Omega, \mathbb{P})$.
- Financial market: riskless asset ≡ 1, and $m \leq d$ risky assets:

$$dS_t^i = S_t^i\Big(b_t^i\, dt + \sum_{j=1}^{d} \sigma_t^{ij} dB_t^j\Big), \qquad i = 1, ..., m.$$

- $|b_t^i(\omega) - b_t^i(\tilde{\omega})| \leq L \sum_{j=1}^d \sup_{s \leq t} |\omega_s^j - \tilde{\omega}_s^j|$, same for σ^{ij}, σ bdd

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- λ_t^i: proportion of an agent's wealth invested in the i^{th} stock at time t: assume $\lambda_t^i \in [0, 1]$ (**no short-selling**)
- $\mathcal{A}(\mathcal{F}^B)$: set of admissible portfolios for the agent without anticipative information ($\mathcal{F}^B$-progressively measurable λ)
- $\mathcal{A}(\mathcal{F}^{B,G})$: set of admissible portfolios for the agent wit anticipative information ($\mathcal{F}^{B,G}$-progressively measurable λ)

Utility maximisation

→ We want to compare the utility maximization problems:

$$v = \sup_{\lambda \in \mathcal{A}(\mathcal{F}^B)} \mathbb{E}[U(X_T^\lambda)], \qquad v(G) = \sup_{\lambda \in \mathcal{A}(\mathcal{F}^{B,G})} \mathbb{E}[U(X_T^\lambda)].$$

- $(X_t^\lambda)_t$: wealth process corresponding to λ, $X_0^\lambda = 1$.
- utility function $U : \mathbb{R}_+ \to \mathbb{R}$ concave, increasing, and s.t.

$$F := U \circ exp \quad \text{is C-Lipschitz, concave and increasing.}$$

e.g. $U(x) = \frac{x^a}{a}$, $a \leq 0$; $U(x) = ln(x)$; $U(x) = -\frac{1}{a} e^{-ax}$, $a \geq 1$

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Proposition

Assume that v and v(G) are both finite, then

$$0 \leq v(G) - v \leq K \inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\gamma)} \mathbb{E}^\pi[V_T(\bar{\omega} - \omega)].$$

Utility maximisation

Remark. In a complete market, for log utility, and for initial enlargements of filtrations, the difference $v(G) - v$ is known explicitly (Pikovsky-Karatzas 1996).

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- $(\pi, \mathcal{F} \times \{\emptyset, C\})$-optional projection: $\tilde{\lambda} \in \mathcal{A}(\mathcal{F}^B)$
- in particular $\tilde{\lambda}_t(\omega) = \tilde{\lambda}_t(\omega,\bar{\omega}) = \mathbb{E}^\pi[\hat{\lambda}_t|\mathcal{F}_t] = \mathbb{E}^\pi[\hat{\lambda}_t|\mathcal{F}_T]$
- substitute in v

Optimal stopping

In spirit same as in the above problems, but "projecting stopping times" is less obvious. We need a weaker notion:

Definition (Randomized stopping time (RST))

A RST Σ w.r.to a filtration $\mathcal{H}$, written $\Sigma \in RST(\mathcal{H})$, is a càdlàg, increasing, $\mathcal{H}$-adapted process, with $\Sigma_0 = 0$ and $\Sigma_T = 1$.

Remark. Projection of RSTs is possible thanks to our proposition on projections under causality.

Optimal stopping

Some motivation

- We want to minimize $\mathbb{E}[\ell(B,\tau)]$, where $\ell : C[0,T] \times \mathbb{R}_+$ is a given cost function, B a Brownian motion on some probability space, and τ a stopping time.
- An analogous formulation of the problem can be done by fixing the probability space to be the path space $C[0,T]$, endowed with Wiener measure and with canonical process W.
- The cost to be paid for having fixed the probability space is that optimization should be done over randomized stopping times instead of stopping times.

Optimal stopping

→ We want to compare the optimization problems:

$$v := \inf_{L \in RST(\mathcal{F})} \mathbb{E}\left[\int \ell(W,t) dL_t\right], \quad v(G) := \inf_{L \in RST(\mathcal{G})} \mathbb{E}\left[\int \ell(W,t) dL_t\right].$$

- The cost function ℓ is $\mathcal{F}$-optional, and K-Lipschitz in its first argument with respect to a metric d on $\mathcal{C} \times \mathcal{C}$, uniformly in time

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Proposition

Assume that v and v(G) are both finite, then

$$0 \leq v - v(G) \leq K \inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\gamma)} \mathbb{E}^{\pi}[d(\omega, \bar{\omega})].$$

E.g. $\ell(x,t) = f(x_t)$ and $\ell(x,t) = f(\sup_{s \leq t} x_s)$ satisfy the above assumption, with $d(\omega, \tilde{\omega}) = \|\omega - \tilde{\omega}\|_\infty$, if f is Lipschitz. In this case

$$0 \leq v - v(G) \leq K \inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\gamma)} \mathbb{E}^{\pi}[V_T(\bar{\omega} - \omega)].$$

Conclusions

- We impose causal constraint on transport plans
- We show attainability and duality for the causal optimal transport problem
- We characterize the preservation of semimartingale property in enlarged filtrations via causal optimal transport problems
- We use causal transport to estimate the value of additional information in several stochastic optimization problems

Bibliography

- T. Yamada and S. Watanabe, “On the uniqueness of solutions of stochastic differential equations”, Journal of Mathematics of Kyoto University 11/1, 155-167, 1971
- J. Jacod “Weak and strong solutions of stochastic differential equations”, Stochastics, 3/3, 171–191, 1980
- R. Lassalle, “Causal transference plans and their Monge-Kantorovich problems”, preprint, 2015
- G. Pflug, “Version-independence and nested distributions in multistage stochastic optimization”, SIAM Journal on Optimization 20/3, 1406-1420, 2009
- B. Acciaio, J. Backhoff and A. Zalashko, “Causal optimal transport and its links to enlargement of filtrations and stochastic optimal control”, working paper, 2016

THANK YOU FOR YOUR ATTENTION!