Quasiconvex set-valued risk measures: compositions and duality theory

Çağın Ararat
Bilkent University, Ankara

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University of Southern California

Multivariate risk measures

- Risk measures for random vectors
  - Multi-asset markets with frictions: Jouini, Meddeb, Touzi '04; Hamel, Heyde, Rudloff '11; Cascos, Molchanov '16
  - Networks of financial institutions: Chen, Iyengar, Moallemi '13; Feinstein, Rudloff, Weber '17; Biagini, Fouque, Frittelli, Meyer-Brandis '18

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  - Utility-based shortfall risk measures: A., Hamel, Rudloff '17; Armenti, Crépey, Drapeau, Papapantoleon '18
  - Market risk measures: Hamel, Rudloff, Yankova '13; A., Hamel, Rudloff '17
  - Systemic risk measures
  - Time-consistent dynamic risk measures: Feinstein, Rudloff '13; '15

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  - Time-consistent dynamic risk measures: Feinstein, Rudloff '13; '15
- Observation: Compositions of set-valued functions show up in all these settings!

Systemic risk

- Interconnected financial system
- Failures affecting multiple entities
  - e.g. chain of defaults
- Important in the event of financial crisis
- Systemic vs. institutional risk

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- Look for capital allocations $y \in \mathbb{R}^d$ that are "inserted" to the system before the shock is realized in such a way that the system becomes safe enough.
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- Examples:
  - Acceptance sets that are insensitive to capital levels (CIM '13):

$$\mathcal{A}(y) = \left\{ X \in L_d^0 \mid \Lambda \circ X + \sum_{i=1}^{d} y_i \in \mathcal{A}_\rho \right\},$$

where $\Lambda \colon \mathbb{R}^d \to \mathbb{R}$ is an aggregation function (increasing, concave) and $\mathcal{A}_\rho = \{\rho \leq 0\}$ is the acceptance of a scalar convex risk measure ρ.

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  - Acceptance sets that are sensitive to capital levels (FBW '17, BFFM '18, AR '18):

$$\mathcal{A}(y) = \left\{ X \in L_d^0 \mid \Lambda \circ (X + y) \in \mathcal{A}_\rho \right\}.$$

- More examples:
  - Acceptance sets of shortfall risk measures (AHR '17):

$$\mathcal{A}(y) = \left\{ X \in L_d^0 \mid \mathbb{E}\left[u \circ (X + y)\right] \in C \right\},$$

where $u\colon \mathbb{R}^d \to \mathbb{R}^d$ is a vector-valued utility function and $C \subseteq \mathbb{R}^d$ is a set such that $C + \mathbb{R}^d_+ = C$.

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  - Nonlinear interaction with capital levels (BFFM '18):

    $$\mathcal{A}(y) = \left\{X \in L_d^0 \mid \Gamma \circ (X, y) \in \mathcal{A}_\rho\right\},$$

    where $\Gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is an aggregation function (e.g. related to Eisenberg-Noe model).

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- Generalization for a lender of last resort: use random capital allocations $Y \in L^0_d$ with a deterministic sum $\sum_{i=1}^d Y_i \in \mathbb{R}$ to be allocated when the shock is realized. Take
$$\mathcal{A}(Y) = \left\{X \in L^0_d \mid \Gamma \circ (X, Y) \in \mathcal{A}_\rho\right\}.$$

- Let $\mathcal{X} = L^0_d$ and $\mathcal{Y}$ the linear space of all capital allocations (e.g. $\mathbb{R}^d$ or a subspace of L^0_d).
- The acceptance family $(\mathcal{A}(Y))_{Y\in\mathcal{Y}}$ can be seen as a set-valued function $\mathcal{A}\colon \mathcal{Y} \to 2^{\mathcal{X}}$. Define $F\colon \mathcal{X} \to 2^{\mathcal{Y}}$ by

$$F(X) = \mathcal{A}^{-1}(X) = \{Y \in \mathcal{Y} \mid X \in \mathcal{A}(Y)\}.$$

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- How to define a systemic risk measure?
  - Fully set-valued approach assuming $\mathcal{Y} = \mathbb{R}^d$ (FRW '17): The systemic risk measure is defined as $R^{\text{sys}} = F$.
  - Scalarization approach using a "price" functional $\pi\colon \mathcal{Y} \to \mathbb{R}$ (BFFM '18): The systemic risk measure is defined by

$$\rho^{\text{sys}}(X) = \inf_{Y\in\mathcal{Y}} \{\pi(Y) \mid X \in \mathcal{A}(Y)\} = \inf_{Y\in\mathcal{Y}} \{\pi(Y) \mid Y \in F(X)\}\,.$$

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    $\to$ too scalar, allocation part is killed

- Let $\mathcal{Z}$ be a linear space of "prices" for capital allocations. (Typically, $\mathcal{Z} = \mathbb{R}^m$ with $1 \leq m \leq d$.)
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  - In general: if $\mathcal{Y}$ is a space of random variables, then G can be a monotone, convex/quasi-convex set-valued function.
- The systemic risk measure can be defined as the set-valued composition of F and G:

$$R^{\mathsf{sys}}(X) = G \circ F(X) = \bigcup_{Y \in F(X)} G(Y).$$

Market risk measures

- d-asset market where trading at a time $t \in \{0, 1, \ldots, T\}$ is subject to transaction costs given by a random convex solvency region K_t
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- Enforce liquidation into the first $m \leq d$ assets. ($\mathcal{Y} = L^0_m$.)
- At time T, her terminal position is in the set

$$F(X) = \left\{Y \in \mathcal{Y} \mid BY \in X - L^0_d(\mathcal{F}_0, K_0) - \ldots - L^0_d(\mathcal{F}_T, K_T)\right\},$$

where $By = (y_1, \ldots, y_m, 0, \ldots, 0)$.

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- Let $\mathcal{Z} = \mathbb{R}^m$. The risk of the terminal position is evaluated by a set-valued risk measure $G\colon \mathcal{Y} \to 2^{\mathcal{Z}}$.
- The market risk of the investor is defined as the "least" achievable risk by trading in the market (AHR '17):

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- When G is convex and translative (usual risk measure properties), R^{mar} can be seen as a set-valued infimal convolution, which yields nice duality results.

Set-valued compositions

- Questions of interest for a set-valued composition $G \circ F$:
  - When is it monotone?
  - When is it convex/quasi-convex?
  - What topological properties should it have?
  - What is the dual representation of it in terms of the dual representations of F, G?
- Our focus here: more general quasiconvex (not-necessarily translative) case.

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- Let $(\mathcal{X}, \leq), (\mathcal{Y}, \leq)$ be preordered linear spaces with respective cones $\mathcal{X}_+, \mathcal{Y}_+$ of positive elements.

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- $\mathcal{P}_+(\mathcal{Y})$: set of all $\mathcal{Y}_+$-monotone subsets of $\mathcal{Y}$, order-complete lattice w.r.t. $\supseteq$:

$$\inf \mathcal{D} = \bigcup_{D \in \mathcal{D}} D, \quad \sup \mathcal{D} = \bigcap_{D \in \mathcal{D}} D.$$

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  $\rightarrow$ Higher return reduces risk.

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- Similar definition for F^{-1}. $\to$ More positions become acceptable as capital allocation increases.
- $F(X) \in \mathcal{P}_+(\mathcal{Y})$ for every $X \in \mathcal{X}$ if and only if F^{-1} is decreasing.
  F is decreasing if and only if $F^{-1}(Y) \in \mathcal{P}_+(\mathcal{X})$ for every $Y \in \mathcal{Y}$.

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- Set-valued composition of F and G:

$$G \circ F(X) = \bigcup_{Y \in F(X)} G(Y).$$

- Inverse:

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- Since G has monotone values, $G \circ F$ has monotone-values.
- Since F is decreasing, $G \circ F$ is decreasing.

- F is said to be quasiconvex if $F(\lambda X^1 + (1-\lambda)X^2) \supseteq F(X^1) \cap F(X^2)$ for every $X^1, X^2 \in \mathcal{X}$ and $\lambda \in [0,1]$.
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- F is convex if and only if $\operatorname{gr} F := \{(X,Y) \mid Y \in F(X)\}$ is convex if and only if $\operatorname{gr} F^{-1}$ is convex if and only if F^{-1} is convex.
- Convexity implies quasiconvexity as well as convex-valuedness.

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  - One can directly start working on the duality of $G \circ F$ with suitable topological assumptions.
- Bad news: If F and G are quasiconvex, then $G \circ F$ may fail to be quasiconvex. ☹

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- Bad news: If F and G are quasiconvex, then $G \circ F$ may fail to be quasiconvex. ☹
  - One remedy: Assume F is convex.
  - Any weaker assumption?

Set-valued compositions: convexity properties

- $F\colon \mathcal{X} \to \mathcal{P}_+(\mathcal{Y})$ decreasing, $G\colon \mathcal{Y} \to \mathcal{P}_+(\mathcal{Z})$ increasing.
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  - Apparently, this property is used in BFFM '18 as a "no-name" property.

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- Any intuitive characterization of natural quasiconvexity?
- Suppose $\mathcal{Y}$ is locally convex with topological dual $\mathcal{Y}^*$. Define the support function of $F(X)$ at $Y^* \in \mathcal{Y}^*$ by

$$\sigma_{F(X)}(Y^*) := \inf_{Y \in F(X)} \langle Y^*, Y \rangle .$$

If $F(X) \neq \emptyset$, then $\sigma_{F(X)}(Y^*) = -\infty$ for every $Y^* \in \mathcal{Y}^* \backslash (\mathcal{Y}_+)^+$, where $(\mathcal{Y}_+)^+$ is the positive dual cone of $\mathcal{Y}_+$.

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- Characterization: Suppose that F has closed and convex values. Then, F is naturally quasiconvex if and only if the function $X \mapsto \sigma_{F(X)}(Y^*)$ on $\mathcal{X}$ is quasiconvex for every $Y^* \in (\mathcal{Y}_+)^+$.

Set-valued functions/compositions: program

- Question 1: Study the dual representation of F when it is ...
  - quasiconvex: Drapeau, Hamel, Kupper '16
  - naturally quasiconvex: now!
  - convex: Hamel '09
- Question 2: Study the dual representation of $G \circ F$ when ...
  - F is naturally quasiconvex and G is quasiconvex,
  - F is naturally quasiconvex and G is naturally quasiconvex,
  - F is convex and G is convex. (For scalar functions: Zalinescu '02; Boț, Grad, Wanka '09)

Set-valued functions: quasiconvex case

- F is said to be lower-level closed if $F^{-1}(Y)$ is a closed set for every $Y \in \mathcal{Y}$.

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- Dual representation (DHK '16): Suppose $F\colon \mathcal{X} \to \mathcal{P}_+(\mathcal{Y})$ is a decreasing quasiconvex lower-level closed function. Then,

$$F(X) = \bigcap_{X^* \in \mathcal{X}^*} \left\{ Y \in \mathcal{Y} \mid \langle X^*, X \rangle \geq \alpha(Y; X^*) \right\},$$

where α is the penalty function defined by

$$\alpha(Y; X^*) = \sigma_{F^{-1}(Y)}(X^*) = \inf_{X \in F^{-1}(Y)} \langle X^*, X \rangle .$$

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- Special case: $\mathcal{X} = L_d^\infty$ with the weak*-topology, F is a set-valued risk measure (decreasing, quasiconvex), one can pass to vector probability measures $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d) \in \mathcal{M}_d(\mathbb{P})$ and weight vectors $w \in \mathbb{R}^d_+$ in the dual representation:

$$F(X) = \bigcap_{\mathbb{Q} \in \mathcal{M}_d(\mathbb{P}), w \in \mathbb{R}^d_+ \setminus \{0\}} \left\{Y \in \mathcal{Y} \mid w^\mathsf{T} \mathbb{E}^{\mathbb{Q}}[X] \geq \tilde{\alpha}(Y; \mathbb{Q}, w)\right\},$$

  where

$$\tilde{\alpha}(Y; \mathbb{Q}, w) = \inf_{X \in F^{-1}(Y)} w^\mathsf{T} \mathbb{E}^{\mathbb{Q}}[X] .$$

Set-valued functions: naturally quasiconvex case

- F is said to be upper demicontinuous if for every $X^0 \in \mathcal{X}$ and open halfspace $H \subseteq \mathcal{Y}$ with $F(X^0) \subseteq H$, there exists a neighborhood V of X^0 such that $F(X) \subseteq H$ for every $X \in V$.

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- If F is upper demicontinuous, then the function $X \mapsto \sigma_{F(X)}(Y^*)$ on $\mathcal{X}$ is lower semicontinuous for every $Y^* \in (\mathcal{Y}_+)^+$ (which implies that F is lower-level closed).
- The converse holds if the infimum in the definition of $\sigma_{F(X)}(Y^*)$ is attained for every $X \in \mathcal{X}$ and $Y^* \in (\mathcal{Y}_+)^+$ (e.g. if $F(X) = \tilde{F}(X) + \mathcal{X}_+$ for some compact-valued $\tilde{F}$).

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$$F(X) = \bigcap_{X^* \in \mathcal{X}^*, Y^* \in (\mathcal{Y}_+)^+} \{Y \in \mathcal{Y} \mid \langle X^*, X \rangle \geq \beta(X^*, Y^*, \langle Y^*, Y \rangle)\},$$

where β is the natural penalty function defined by

$$\beta(X^*, Y^*, r) = \inf_{X \in \mathcal{X}} \left\{ \langle X^*, X \rangle \mid r \geq \sigma_{F(X)}(Y^*) \right\}.$$

Set-valued functions: naturally quasiconvex case

- Special case: $\mathcal{X} = L_d^\infty, \mathcal{Y} = \mathbb{R}^d$

$$F(X) = \bigcap_{\mathbb{Q}\in\mathcal{M}_d(\mathbb{P}), w, c\in\mathbb{R}^d_+\backslash\{0\}} \left\{ y \in \mathbb{R}^d \mid w^\mathsf{T}\mathbb{E}^\mathbb{Q}[X] \geq \tilde{\beta}(\mathbb{Q}, w, c, c^\mathsf{T} y) \right\},$$

where

$$\tilde{\beta}(\mathbb{Q}, w, c, r) = \inf_{X\in L_d^\infty} \left\{ w^\mathsf{T}\mathbb{E}^\mathbb{Q}[X] \mid r \geq \inf_{y'\in F(X)} c^\mathsf{T} y' \right\}.$$

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$$F(X) = \bigcap_{\substack{\mathbb{Q} \in \mathcal{M}_d(\mathbb{P}), \mathbb{S} \in \mathcal{M}_m(\mathbb{P}), \\ w \in \mathbb{R}_+^d \setminus \{0\}, c \in \mathbb{R}_+^m \setminus \{0\}}} \left\{ Y \in L_m^\infty \mid w^\mathsf{T} \mathbb{E}^\mathbb{Q}[X] \geq \tilde{\beta}(\mathbb{Q}, \mathbb{S}, w, c, c^\mathsf{T} \mathbb{E}^\mathbb{S}[Y]) \right\},$$

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Set-valued functions: convex case

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- Dual representation (H '09): Suppose $F\colon \mathcal{X} \to \mathcal{P}_+(\mathcal{Y})$ is a decreasing convex closed function. Then,

$$F(X) = \bigcap_{X^* \in \mathcal{X}^*, Y^* \in (\mathcal{Y}_+)^+} \left\{ Y \in \mathcal{Y} \mid \langle Y^*, Y \rangle \geq \gamma(X^*, Y^*) - \langle X^*, X \rangle \right\},$$

where γ is the support function of $\operatorname{gr} F$ defined by

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$$F(X) = \bigcap_{\mathbb{Q} \in \mathcal{M}_d(\mathbb{P}); w, c \in \mathbb{R}^d_+ \setminus \{0\}} \left\{ y \in \mathbb{R}^d \mid c^\mathsf{T} y \geq \tilde{\gamma}(\mathbb{Q}, w, c) - w^\mathsf{T} \mathbb{E}^\mathbb{Q}[X] \right\},$$

where $\tilde{\gamma}$ is the conjugate function of the support function of the acceptance family defined by

$$\tilde{\gamma}(\mathbb{Q}, w, c) = \inf_{y \in \mathbb{R}^d} \left(c^\mathsf{T} y + \inf_{X \in F^{-1}(y)} w^\mathsf{T} \mathbb{E}^\mathbb{Q}[X] \right).$$

Set-valued compositions: quasiconvex case

- Suppose $F\colon \mathcal{X} \to \mathcal{P}_+(\mathcal{Y})$ is a decreasing naturally quasiconvex upper demicontinuous with closed convex values. Let β_F be its natural penalty function.
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where

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In other words,

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Set-valued compositions: naturally quasiconvex case

- Suppose $F\colon \mathcal{X} \to \mathcal{P}_+(\mathcal{Y}), G\colon \mathcal{Y} \to \mathcal{P}_+(\mathcal{Z})$ are naturally quasiconvex upper demicontinuous lower-level closed functions with closed convex values. Suppose F is decreasing and G is increasing. Let β_F, β_G be their respective natural penalty functions.

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Set-valued compositions: convex case

- Suppose $F\colon \mathcal{X} \to \mathcal{P}_+(\mathcal{Y}), G\colon \mathcal{Y} \to \mathcal{P}_+(\mathcal{Z})$ be convex closed functions. Suppose F is decreasing and G is increasing. Let γ_F, γ_G be their respective support functions of their graphs.

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Example 1: convex scalarizations of set-valued convex risk measures

- $\mathcal{X} = L_d^\infty$, $\mathcal{Y} = \mathbb{R}^d$, $\mathcal{Z} = \mathbb{R}$
- $F\colon L_d^\infty \to \mathcal{P}_+(\mathbb{R}^d)$ decreasing convex closed translative with support function $\tilde{\gamma}$ and acceptance set $\mathcal{A} = F^{-1}(0)$:

$$F(X) = \bigcap_{\mathbb{Q}\in\mathcal{M}_d(\mathbb{P}),\, w\in\mathbb{R}^d_+\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[-X\right] + \left\{y \in \mathbb{R}^d \mid w^{\mathsf{T}} y \geq \tilde{\gamma}(\mathbb{Q}, w)\right\},$$

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- $\pi\colon \mathbb{R}^d \to \mathbb{R}$ increasing convex scalarization function with convex conjugate π^*:

$$\pi^*(w) = \sup_{y\in\mathbb{R}^d} \left(w^{\mathsf{T}} y - \pi(y)\right).$$

- Scalarized risk measure (e.g. as a systemic risk measure, convex but not translative in general):

$$\rho^\pi(X) = \inf\left\{\pi(y) \mid y \in F(X)\right\} = \inf\left\{\pi(y) \mid X + y \in \mathcal{A}\right\}$$

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- Dual representation:

$$\rho^\pi(X) = \sup_{\mathbb{Q}\in\mathcal{M}_d(\mathbb{P}),\, w\in\mathbb{R}^d_+\setminus\{0\}} \left(w^{\mathsf{T}} \mathbb{E}^{\mathbb{Q}}[-X] + \tilde{\gamma}(\mathbb{Q}, w) - \pi^*(w) \right)$$

- Special case: F sensitive systemic risk measure:

$$F(X) = \left\{ y \in \mathbb{R}^d \mid \Lambda \circ (X + y) \in \mathcal{A}_\rho \right\},$$

where $\Lambda \colon \mathbb{R}^d \to \mathbb{R}$ is the aggregation function (determined by the network model) and $\mathcal{A}_\rho$ is the acceptance set of a scalar convex risk measure ρ with penalty function α.

- $g(z) = \sup_{y \in \mathbb{R}^d} \left(\Lambda(y) - z^\mathsf{T} y \right)$: convex conjugate of $y \mapsto -\Lambda(-y)$

Example 1: convex scalarizations of set-valued convex risk measures

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- $g(z) = \sup_{y \in \mathbb{R}^d} \left(\Lambda(y) - z^\mathsf{T} y \right)$: convex conjugate of $y \mapsto -\Lambda(-y)$
- Scalarized systemic risk measure:

$$\rho^\pi(X) = \inf \left\{ \pi(y) \mid \Lambda \circ (X + Y) \in \mathcal{A}_\rho \right\}$$

Example 1: convex scalarizations of set-valued convex risk measures

- Special case: F sensitive systemic risk measure:

$$F(X) = \left\{ y \in \mathbb{R}^d \mid \Lambda \circ (X + y) \in \mathcal{A}_\rho \right\},$$

where $\Lambda\colon \mathbb{R}^d \to \mathbb{R}$ is the aggregation function (determined by the network model) and $\mathcal{A}_\rho$ is the acceptance set of a scalar convex risk measure ρ with penalty function α.

- $g(z) = \sup_{y \in \mathbb{R}^d} \left(\Lambda(y) - z^\mathsf{T} y \right)$: convex conjugate of $y \mapsto -\Lambda(-y)$
- Scalarized systemic risk measure:

$$\rho^\pi(X) = \inf \left\{ \pi(y) \mid \Lambda \circ (X + Y) \in \mathcal{A}_\rho \right\}$$

- Dual representation:

$$\rho^\pi(X) = \sup_{\substack{\mathbb{Q} \in \mathcal{M}_d(\mathbb{P}),\\ \mathbb{S} \in \mathcal{M}^e(\mathbb{P}),\\ w \in \mathbb{R}^d_+ \setminus \{0\},\\ \lambda > 0}} \left(w^\mathsf{T} \mathbb{E}^\mathbb{Q}[-X] - \lambda \mathbb{E}^\mathbb{S}\left[g \circ \left(\frac{w}{\lambda} \cdot \frac{d\mathbb{Q}}{d\mathbb{S}} \right) \right] - \lambda \alpha(\mathbb{S}) - \pi^*(w) \right),$$

Example 2: systemic risk measure with random capital allocations

- random shocks $\mathcal{X} = L_d^\infty$, capital allocations $\mathcal{Y} = \left\{ Y \in L_d^\infty \mid \sum_{i=1}^d Y_i \in \mathbb{R} \right\}$
- price of allocation Y: $\pi(Y) = \sum_{i=1}^d Y_i$
- aggregation function $\Lambda\colon \mathbb{R}^d \to \mathbb{R}$ with $g(z) = \sup_{y \in \mathbb{R}^d} \left(\Lambda(y) - z^\mathsf{T} y \right)$, the convex conjugate of $y \mapsto -\Lambda(-y)$

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- quasiconvex scalar risk measure $\rho \colon \mathcal{Y} \to (-\infty, +\infty]$ with acceptance set family $(B(z))_{z \in \mathbb{R}}$
- Scalarized systemic risk measure:

$$\rho^{\mathsf{sys}}(X) = \inf \left\{ \sum_{i=1}^d Y_i \mid \Lambda \circ (X + Y) \in B\left(\sum_{i=1}^d Y_i\right) \right\}$$

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- Dual representation:

$$\rho^{\mathsf{sys}}(X) = \sup_{\substack{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P}), \\ \mathbb{S} \in \mathcal{M}^e(\mathbb{P}), \\ w \in [0,1], \\ \lambda > 0}} \left(w \sum_{i=1}^d \mathbb{E}^{\mathbb{Q}}\left[-X_i \right] - \lambda \mathbb{E}^{\mathbb{S}} \left[g \circ \left(\frac{w}{\lambda} \frac{d\mathbb{Q}}{d\mathbb{S}} \mathbb{1}_d \right) \right] + \inf_{z \in \mathbb{R}} \left(z^+ + \lambda \alpha(\mathbb{S}; z) \right) \right),$$

where α is the penalty function of ρ given by

$$\alpha(\mathbb{S}; z) = \inf \left\{ \mathbb{E}^{\mathbb{S}}\left[U \right] \mid U \in B(z) \right\}.$$

To Do

- More concrete examples of systemic risk
- Time-consistent dynamic set-valued naturally quasiconvex risk measures:

$$R_s(X) = R_s \circ -R_t(X) = \bigcup_{Y \in R_t(X)} R_s(-Y)$$

  for every $0 \leq s \leq t \leq T$.
- Dual characterization of time-consistency?

Thank you!

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