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Incomplete-Market Equilibria with a Large Number of Heterogeneous Agents and BSDEs

USC | Department of Mathematics

Monday, September 23, 2019

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Principle of Maximum for Games with a Large Number of Players in Discrete Time, and Some Concrete Macroeconomic Models

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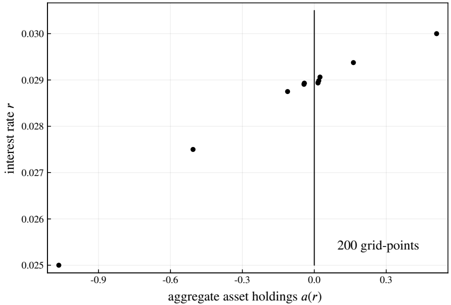

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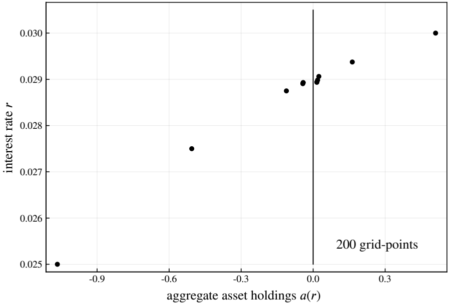

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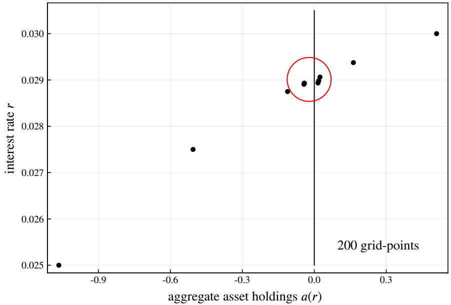

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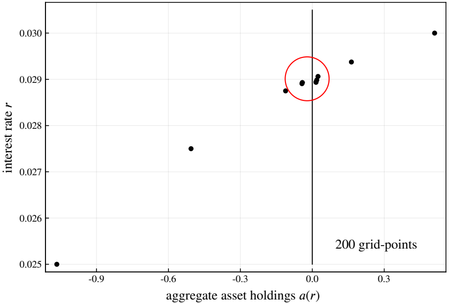

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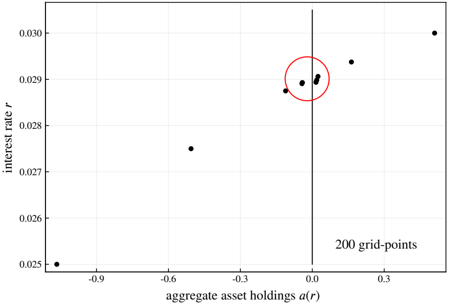

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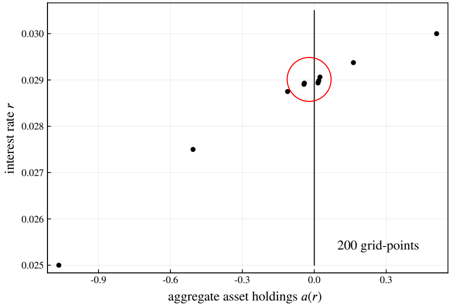

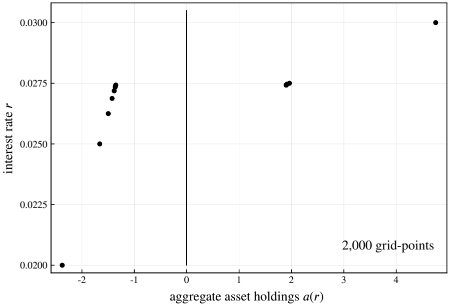

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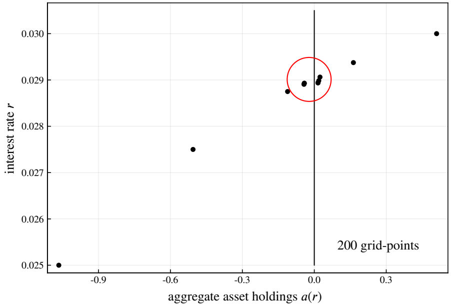

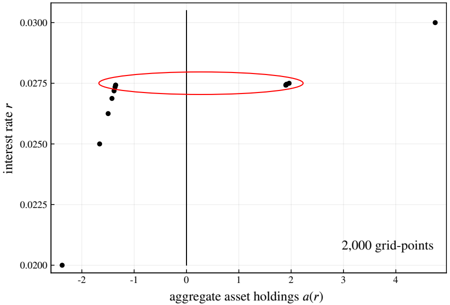

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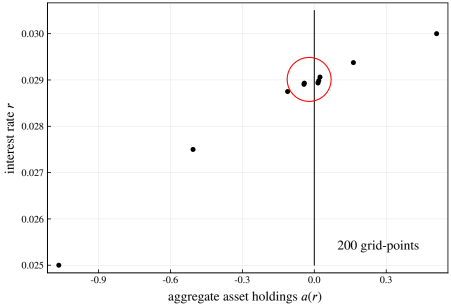

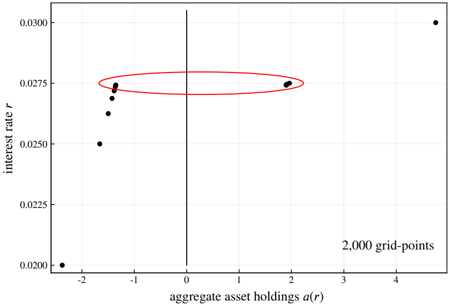

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λ_∞ is then re-interpreted as the long run cross-sectional distribution of households over $\mathcal{A} \times \mathcal{S}$

and the market clearing is $a(r) \stackrel{\text{def}}{=} \sum\nolimits_{a \in \mathcal{A},\, s \in \mathcal{S}} g(a, s) \lambda_\infty(a, s) = 0$.

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The idea is to find the zero of the function $r \rightsquigarrow a(r)$.

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Let us examine briefly the solution procedure from Ch. 18 in L&S. The employment hours (per period) fluctuate inside $\mathcal{S} = \left(e^{-1.2}, e^{-0.8}, e^{-0.4}, e^{0}, e^{0.4}, e^{0.8}, e^{1.2} \right) \in \mathbb{R}^7_{++}$ with transition matrix $\mathcal{P}_s(\sigma)$, $s, \sigma \in \mathcal{S}$. A generic ("atomistic") household is faced with the (private) Bellman equation

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where a_{t+1} is the amount invested in IOU during period t, and $\mathcal{A} \subset \mathbb{R}$ is a fixed finite grid.

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N.B. If the agents can sample their individual state (a, s) from the law λ_∞ – *independently from one another* – then the cross-sectional distribution will indeed be λ_∞.

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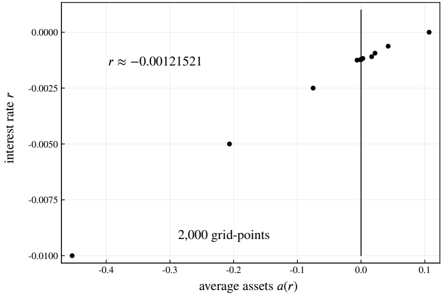

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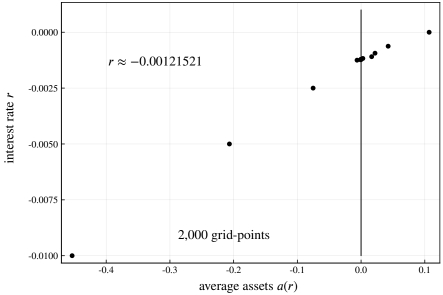

N.B. There is a continuous-time version: Achdou et al. (2014) Phil. Trans. R. Soc. A 372: 20130397

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What follows is an extension of (Dumas & L 2012) for the case infinitely many agents, with a bridge to mean field games and control.

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Assume a constant-return-to-scale production function with stochastic TFP:

$$F_{t+1}(K_t, N) = \Xi_{t+1} K_t^{\alpha} N^{1-\alpha}$$

with $K_t \stackrel{\text{def}}{=}$ the aggregate capital stock installed at time t.

The Dual (Principle of Maximum) Approach to GEI

$\mathcal{H} \stackrel{\text{def}}{=}$ the collection of economic agents, with $L = |\mathcal{H}|$ assumed to be "very large."

Time is discrete $t \in \mathbb{N}_T \stackrel{\text{def}}{=} \{0, 1, \ldots, T\}$, and the economic agents are ex-ante identical, with a common utility $\mathbb{R}_{++} \ni c \rightsquigarrow U(c) \in \mathbb{R}$, which is as nice as needed.

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N.B. The rental rates for labor and capital are:

$$w_{t+1} = \Xi_{t+1}(1-\alpha)(K_t/N)^{\alpha} \quad \text{and} \quad \rho_{t+1} = \Xi_{t+1}\alpha(K_t/N)^{\alpha-1}.$$

The Dual (Principle of Maximum) Approach to GEI

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The Dual (Principle of Maximum) Approach to GEI

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The collective state of the population can be described as an element $F \in \mathbb{F}^{\mathcal{S}}$, i.e., as a family of CDFs on the consumption space: $\mathbb{R}_{++} \ni c \rightsquigarrow F_s(c) \in [0,1]\,,\quad s \in \mathcal{S}\,.$

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In every aggregate state $x \in \mathcal{X}$, the choice of a consumption record, $c \in \mathbb{R}_{++}$, for an agent in employment state $s \in \mathcal{S}$, together with the choice of a collective (consumption) state $F \in \mathbb{F}^{\mathcal{S}}$, completely determines that agent's exiting portfolio record and their next period consumption record, contingent upon the next period realizations of the aggregate state $\xi \in \mathcal{X}$ and the employment state $\sigma \in \mathcal{S}$.

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In every aggregate state $x \in \mathcal{X}$, the choice of a consumption record, $c \in \mathbb{R}_{++}$, for an agent in employment state $s \in \mathcal{S}$, together with the choice of a collective (consumption) state $F \in \mathbb{F}^{\mathcal{S}}$, completely determines that agent's exiting portfolio record and their next period consumption record, contingent upon the next period realizations of the aggregate state $\xi \in \mathcal{X}$ and the employment state $\sigma \in \mathcal{S}$.

The portfolio record $\{\theta_{t,x,s}(c,F), \vartheta_{t,x,s}(c,F)\}$ corresponds to an actual investment in the bond of $\theta_{t,x,s}(c,F)/L$ and an actual investment in the capital stock of $\vartheta_{t,x,s}(c,F)/L$. The next period consumption record $v^{\xi,\sigma}_{t,x,s}(c,F)$ corresponds to an actual consumption level of $v^{\xi,\sigma}_{t,x,s}(c,F)/L$.

The Dual (Principle of Maximum) Approach to GEI

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(b) a collection of assignments (representing risk-free rates and installed productive capital)

$$\mathcal{X} \times \mathbb{F}^{\mathcal{S}} \ni (x, \digamma) \quad \rightsquigarrow \quad r_t(x, \digamma) \in \mathbb{R}\,, \quad K_t(x, \digamma) \in \mathbb{R}_{++}\,, \quad \text{for all } t \in \mathbb{N}_{T-1}\,;$$

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(c) the period $(t + 1)$ consumption record for the same agent, contingent upon the realizations of $\xi \in \mathcal{X}$ and $\sigma \in \mathcal{S}$, is $v^{\xi,\sigma}_{t,x,s}(c, \digamma)$;

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(a) the cross-sectional distribution of agents over the consumption space at the end of period t is F;

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and (d) the markets clear:

$$\sum_{s \in \mathcal{S}} \pi(s) \int_0^\infty \theta_{t,x,s}(c, F)\, \mathrm{d}F_s(c) = 0 \qquad \text{and} \qquad \sum_{s \in \mathcal{S}} \pi(s) \int_0^\infty \vartheta_{t,x,s}(c, F)\, \mathrm{d}F_s(c) = K_t(x, F).$$

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(a) the cross-sectional distribution of agents over the consumption space at the end of period t is $\digamma$;

(b) the exiting portfolio record for an agent who is in employment state $s \in \mathcal{S}$ and has chosen (as a rational price taker) the consumption record $c \in \mathbb{R}_{++}$ is $\{\theta_{t,x,s}(c,\digamma), \vartheta_{t,x,s}(c,\digamma)\}$;

(c) the period $(t+1)$ consumption record for the same agent, contingent upon the realizations of $\xi \in \mathcal{X}$ and $\sigma \in \mathcal{S}$, is $\nu_{t,x,s}^{\xi,\sigma}(c,\digamma)$;

and (d) the markets clear:

$$\sum_{s\in\mathcal{S}} \pi(s) \int_0^\infty \theta_{t,x,s}(c,\digamma)\,\mathrm{d}\digamma_s(c) = 0 \qquad \text{and} \qquad \sum_{s\in\mathcal{S}} \pi(s) \int_0^\infty \vartheta_{t,x,s}(c,\digamma)\,\mathrm{d}\digamma_s(c) = K_t(x,\digamma).$$

The Dual (Principle of Maximum) Approach to GEI

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At time $t \in \mathbb{N}_{T-1}$ an agent would enter the employment state $s \in \mathcal{S}$ with wealth z/L for

$$z \stackrel{\text{def}}{=} \theta_{t-1} r_{t-1} + \vartheta_{t-1} \rho_t + w_t s \,,$$

which amount is treated as a given resource.

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which amount is treated as a given resource.

Given any $x \in \mathcal{X}$ and any $\digamma \in \mathbb{F}^{\mathcal{S}}$, the agent's value, $V_{t,x,\digamma_s}(z/L)$, obtains from the optimization problem

$$\underset{c,\theta,\vartheta,Z(\xi,\sigma)}{\text{maximize}}\Big(U(c/L) + \beta \sum_{\xi\in\mathcal{X}} \sum_{\sigma\in\mathcal{S}} V_{t+1,\xi,\tilde{\digamma},\sigma}\big(Z(\xi,\sigma)/L\big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\Big)\,,$$

subject to:

$$Z(\xi,\sigma) = \theta\big(1 + r_t(x,\digamma)\big) + \vartheta\rho_{t+1}(x,\digamma,\xi) + \sigma w_{t+1}(x,\digamma,\xi)\,, \quad \text{for every } \ \xi\in\mathcal{X}\,,\ \sigma\in\mathcal{S}\,,$$

$$\text{and} \quad \theta + \vartheta + c = z\,.$$

At time $t \in \mathbb{N}_{T-1}$ an agent would enter the employment state $s \in \mathcal{S}$ with wealth z/L for

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Given any $x \in \mathcal{X}$ and any $\digamma \in \mathbb{F}^{\mathcal{S}}$, the agent's value, $V_{t,x,\digamma,s}(z/L)$, obtains from the optimization problem

$$\underset{c,\theta,\vartheta,Z(\xi,\sigma)}{\text{maximize}}\Big(U(c/L) + \beta \sum\nolimits_{\xi\in\mathcal{X}} \sum\nolimits_{\sigma\in\mathcal{S}} V_{t+1,\xi,\tilde{\digamma},\sigma}\big(Z(\xi,\sigma)/L\big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\Big)\,,$$

subject to:

$$Z(\xi,\sigma) = \theta\big(1 + r_t(x,\digamma)\big) + \vartheta\rho_{t+1}(x,\digamma,\xi) + \sigma w_{t+1}(x,\digamma,\xi)\,, \quad \text{for every } \ \xi \in \mathcal{X}\,,\ \sigma \in \mathcal{S}\,,$$
$$\text{and} \quad \theta + \vartheta + c = z\,.$$

N.B. The agent takes the present period $\digamma \in \mathbb{F}^{\mathcal{S}}$ and the next period $\tilde{\digamma} \in \mathbb{F}^{\mathcal{S}}$ as given.

The Dual (Principle of Maximum) Approach to GEI

At time $t \in \mathbb{N}_{T-1}$ an agent would enter the employment state $s \in \mathcal{S}$ with wealth z/L for

$$z \overset{\text{def}}{=} \theta_{t-1} r_{t-1} + \vartheta_{t-1}\rho_t + w_t s\,,$$

which amount is treated as a given resource.

Given any $x \in \mathcal{X}$ and any $F \in \mathbb{F}^{\mathcal{S}}$, the agent's value, $V_{t,x,F,s}(z/L)$, obtains from the optimization problem

$$\underset{c,\theta,\vartheta,Z(\xi,\sigma)}{\text{maximize}}\Big(U(c/L) + \beta \sum_{\xi\in\mathcal{X}} \sum_{\sigma\in\mathcal{S}} V_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\Big)\,,$$

subject to:

$$Z(\xi,\sigma) = \theta\big(1 + r_t(x,F)\big) + \vartheta\rho_{t+1}(x,F,\xi) + \sigma w_{t+1}(x,F,\xi)\,, \quad \text{for every } \xi \in \mathcal{X}\,,\ \sigma \in \mathcal{S}\,,$$
$$\text{and} \quad \theta + \vartheta + c = z\,.$$

N.B. The agent takes the present period $F \in \mathbb{F}^{\mathcal{S}}$ and the next period $\tilde{F} \in \mathbb{F}^{\mathcal{S}}$ as given.

N.B. The aggregate (collectively decided) installed capital in state (x,F) is $K_t(x,F)$ and

$$w_{t+1}(x,F,\xi) = \xi(1-\alpha)\big(K_t(x,F)/N\big)^{\alpha} \quad \text{and} \quad \rho_{t+1}(x,F,\xi) = \xi\alpha\big(K_t(x,F)/N\big)^{\alpha-1}.$$

The Dual (Principle of Maximum) Approach to GEI

At time $t \in \mathbb{N}_{T-1}$ an agent would enter the employment state $s \in \mathcal{S}$ with wealth z/L for

$$z \overset{\text{def}}{=} \theta_{t-1} r_{t-1} + \vartheta_{t-1}\rho_t + w_t s\,,$$

which amount is treated as a given resource.

Given any $x \in \mathcal{X}$ and any $\digamma \in \mathbb{F}^{\mathcal{S}}$, the agent's value, $V_{t,x,\digamma,s}(z/L)$, obtains from the optimization problem

$$\underset{c,\theta,\vartheta,Z(\xi,\sigma)}{\text{maximize}}\Big(U(c/L) + \beta \sum_{\xi\in\mathcal{X}} \sum_{\sigma\in\mathcal{S}} V_{t+1,\xi,\tilde{\digamma},\sigma}\big(Z(\xi,\sigma)/L\big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\Big)\,,$$

subject to:

$$Z(\xi,\sigma) = \theta\big(1 + r_t(x,\digamma)\big) + \vartheta\rho_{t+1}(x,\digamma,\xi) + \sigma w_{t+1}(x,\digamma,\xi)\,, \quad \text{for every } \ \xi\in\mathcal{X}\,,\ \sigma\in\mathcal{S}\,,$$
$$\text{and} \quad \theta + \vartheta + c = z\,.$$

N.B. The agent takes the present period $\digamma \in \mathbb{F}^{\mathcal{S}}$ and the next period $\tilde{\digamma} \in \mathbb{F}^{\mathcal{S}}$ as given.

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$$w_{t+1}(x,\digamma,\xi) = \xi(1-\alpha)\big(K_t(x,\digamma)/N\big)^{\alpha} \quad \text{and} \quad \rho_{t+1}(x,\digamma,\xi) = \xi\alpha\big(K_t(x,\digamma)/N\big)^{\alpha-1}.$$

N.B. The first set of constraints can be cast as:

$$Z(\xi,\sigma) - \theta\big(1 + r_t(x,\digamma)\big) - \vartheta\rho_{t+1}(x,\digamma,\xi) - \sigma w_{t+1}(x,\digamma,\xi) + \theta + \vartheta + c = z\,, \quad \xi\in\mathcal{X}\,,\ \sigma\in\mathcal{S}\,.$$

The Dual (Principle of Maximum) Approach to GEI

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The Dual (Principle of Maximum) Approach to GEI

The Lagrange multipliers (covariables) attached to the period $(t+1)$ constraints we put in the form $\dfrac{\beta}{L}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$ and the one attached to the period t constraint we put in the form φ/L.

The Dual (Principle of Maximum) Approach to GEI

The Lagrange multipliers (covariables) attached to the period $(t+1)$ constraints we put in the form $\dfrac{\beta}{L}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$ and the one attached to the period t constraint we put in the form φ/L.

The Lagrangian function for the period t private optimization problem is:

$$\mathcal{L}(c,\theta,\vartheta,Z(\xi,\sigma),\Phi(\xi,\sigma),\xi\in\mathcal{X},\sigma\in\mathcal{S}) = U(c/L) + \beta\sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} V_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$$

$$+\frac{\beta}{L}\sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\Big(z - Z(\xi,\sigma) + \theta\big(1+r_t(x,F)\big) + \vartheta\rho_{t+1}(x,F,\xi) + \sigma w_{t+1}(x,F,\xi)$$

$$-\theta-\vartheta-c\Big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma) + \frac{\varphi}{L}\big(z-\theta-\vartheta-c\big)\,.$$

The Dual (Principle of Maximum) Approach to GEI

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$$+\frac{\beta}{L}\sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\Big(z - Z(\xi,\sigma) + \theta\big(1+r_t(x,F)\big) + \vartheta\rho_{t+1}(x,F,\xi) + \sigma w_{t+1}(x,F,\xi)$$
$$-\theta-\vartheta-c\Big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma) + \frac{\varphi}{L}\big(z-\theta-\vartheta-c\big)\,.$$

With $\phi \overset{\text{def}}{=} \varphi + \beta\sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$ the FOCs can be cast as:

$$\Phi(\xi,\sigma) = V'_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big)\,,\quad \phi = U'(c/L)\,,$$
$$\phi = \big(1+r_t(x,F)\big)\beta\sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,,$$
$$\phi = \beta\sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\rho_{t+1}(x,F,\xi)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,.$$

The Dual (Principle of Maximum) Approach to GEI

The Lagrange multipliers (covariables) attached to the period $(t+1)$ constraints we put in the form $\dfrac{\beta}{L}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$ and the one attached to the period t constraint we put in the form φ/L.

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$$+\frac{\beta}{L}\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\Big(z - Z(\xi,\sigma) + \theta\big(1+r_t(x,F)\big) + \vartheta\rho_{t+1}(x,F,\xi) + \sigma w_{t+1}(x,F,\xi)$$
$$-\theta-\vartheta-c\Big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma) + \frac{\varphi}{L}\big(z-\theta-\vartheta-c\big)\,.$$

With $\phi \overset{\text{def}}{=} \varphi + \beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$ the FOCs can be cast as:

$$\Phi(\xi,\sigma) = V'_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big)\,,\quad \phi = U'(c/L)\,,$$
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$$\phi = \beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\rho_{t+1}(x,F,\xi)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,.$$

the envelope theorem ⇨ $V'_{t,x,F,s}(z/L) = \varphi + \beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma) = \phi = U'(c/L)\,.$

The Dual (Principle of Maximum) Approach to GEI

The Lagrange multipliers (covariables) attached to the period $(t+1)$ constraints we put in the form $\frac{\beta}{L}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$ and the one attached to the period t constraint we put in the form φ/L.

The Lagrangian function for the period t private optimization problem is:

$$\mathcal{L}(c,\theta,\vartheta,Z(\xi,\sigma),\Phi(\xi,\sigma),\xi\in\mathcal{X},\sigma\in\mathcal{S}) = U(c/L) + \beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} V_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$$
$$+\frac{\beta}{L}\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\Big(z - Z(\xi,\sigma) + \theta\big(1+r_t(x,F)\big) + \vartheta\rho_{t+1}(x,F,\xi) + \sigma w_{t+1}(x,F,\xi)$$
$$-\theta-\vartheta-c\Big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma) + \frac{\varphi}{L}\big(z-\theta-\vartheta-c\big)\,.$$

With $\phi \overset{\text{def}}{=} \varphi + \beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)$ the FOCs can be cast as:

$$\Phi(\xi,\sigma) = V'_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big)\,,\quad \phi = U'(c/L)\,,$$
$$\phi = \big(1+r_t(x,F)\big)\beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,,$$
$$\phi = \beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\rho_{t+1}(x,F,\xi)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,.$$

the envelope theorem ⇨ $V'_{t,x,F,s}(z/L) = \varphi + \beta\sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}}\Phi(\xi,\sigma)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma) = \phi = U'(c/L)\,.$

N.B. consumption ⇦⇨ covariables

The Dual (Principle of Maximum) Approach to GEI

The Dual (Principle of Maximum) Approach to GEI

The Dual (Principle of Maximum) Approach to GEI

Since $\Phi(\xi,\sigma) = V'_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big) = U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)$, the last two FOCs give the kernel conditions :

$$1 = \big(1 + r_t(x,F)\big)\beta \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \frac{U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)}{U'(c/L)} \mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,,$$

$$1 = \beta \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \frac{U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)}{U'(c/L)} \rho_{t+1}(x,F,\xi)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,.$$

The Dual (Principle of Maximum) Approach to GEI

Since $\Phi(\xi,\sigma) = V'_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big) = U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)$, the last two FOCs give the kernel conditions:

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$$1 = \beta \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \frac{U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)}{U'(c/L)} \rho_{t+1}(x,F,\xi)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,.$$

Using power utility (from now on) $\Rightarrow$ $\dfrac{U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)}{U'(c/L)} = U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/c\big)\,.$

The Dual (Principle of Maximum) Approach to GEI

Since $\Phi(\xi,\sigma) = V'_{t+1,\xi,\tilde{F},\sigma}\big(Z(\xi,\sigma)/L\big) = U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)$, the last two FOCs give the kernel conditions:

$$1 = \big(1 + r_t(x,F)\big)\beta \sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \frac{U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)}{U'(c/L)} \mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,,$$

$$1 = \beta \sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \frac{U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)}{U'(c/L)} \rho_{t+1}(x,F,\xi)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,.$$

Using power utility (from now on) ⇨ $\dfrac{U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/L\big)}{U'(c/L)} = U'\big(v^{\xi,\sigma}_{t,x,s}(c,F)/c\big)\,.$

This removes $L = |\mathcal{H}|$ from the picture, and removes the need for passing to the limit as $L \to \infty$, as long as $L \approx \infty$.

The Dual (Principle of Maximum) Approach to GEI

The Dual (Principle of Maximum) Approach to GEI

The Dual (Principle of Maximum) Approach to GEI

At time $t = T$ there is no future $\Rightarrow$ $\theta_{T,x,s}(c, F) \stackrel{\text{def}}{=} 0, \quad \vartheta_{T,x,s}(c, F) \stackrel{\text{def}}{=} 0, \quad \nu^{\xi,\sigma}_{T,x,s}(c, F) \stackrel{\text{def}}{=} 0.$

The Dual (Principle of Maximum) Approach to GEI

At time $t = T$ there is no future $\quad \Rightarrow \quad \theta_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0, \quad \vartheta_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0, \quad \nu^{\xi,\sigma}_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0.$

Suppose that for some fixed $0 \le t < T$ the following demand functions are known

$$\mathbb{R}_{++} \times \mathbb{F}^{\mathcal{S}} \ni (c,\digamma) \quad \leadsto \quad \theta_{t+1,\xi,\sigma}(c,\digamma) \in \mathbb{R}, \quad \vartheta_{t+1,\xi,\sigma}(c,\digamma), \quad \text{for all} \quad \xi \in \mathcal{X} \quad \text{and} \quad \sigma \in \mathcal{S}.$$

The Dual (Principle of Maximum) Approach to GEI

At time $t = T$ there is no future $\Rightarrow$ $\theta_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0, \quad \vartheta_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0, \quad \nu^{\xi,\sigma}_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0.$

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$$\mathbb{R}_{++} \times \mathbb{F}^{\mathcal{S}} \ni (c,\digamma) \quad \rightsquigarrow \quad \theta_{t+1,\xi,\sigma}(c,\digamma) \in \mathbb{R}, \quad \vartheta_{t+1,\xi,\sigma}(c,\digamma), \quad \text{for all} \quad \xi \in \mathcal{X} \quad \text{and} \quad \sigma \in \mathcal{S}.$$

Given $x \in \mathcal{X}$, choose and fix $\digamma \in \mathbb{F}$, and make an ansatz choice for $r_t(x,\digamma) \in \mathbb{R}$ and $K_t(x,\digamma) \in \mathbb{R}_+$.

The Dual (Principle of Maximum) Approach to GEI

At time $t = T$ there is no future $\Rightarrow$ $\theta_{T,x,s}(c, \digamma) \overset{\text{def}}{=} 0$, $\vartheta_{T,x,s}(c, \digamma) \overset{\text{def}}{=} 0$, $v^{\xi,\sigma}_{T,x,s}(c, \digamma) \overset{\text{def}}{=} 0$.

Suppose that for some fixed $0 \le t < T$ the following demand functions are known

$$\mathbb{R}_{++} \times \mathbb{F}^{\mathcal{S}} \ni (c, \digamma) \quad \rightsquigarrow \quad \theta_{t+1,\xi,\sigma}(c, \digamma) \in \mathbb{R}, \quad \vartheta_{t+1,\xi,\sigma}(c, \digamma), \quad \text{for all} \quad \xi \in \mathcal{X} \quad \text{and} \quad \sigma \in \mathcal{S}.$$

Given $x \in \mathcal{X}$, choose and fix $\digamma \in \mathbb{F}$, and make an ansatz choice for $r_t(x, \digamma) \in \mathbb{R}$ and $K_t(x, \digamma) \in \mathbb{R}_+$.

For every fixed $(s, c) \in \mathcal{S} \times \mathbb{R}_{++}$, consider the following system of $2 + |\mathcal{X}| \times |\mathcal{S}|$ equations, parameterized by $\left(\tilde{\digamma}^{\xi} \in \mathbb{F}^{\mathcal{S}}, \xi \in \mathcal{X}\right)$, for the (same exact number of) unknowns $\theta_{t,x,s}(c, \digamma)$, $\vartheta_{t,x,s}(c, \digamma)$, and $v^{\xi,\sigma}_{t,x,s}(c, \digamma)$, $\xi \in \mathcal{X}$, $\sigma \in \mathcal{S}$:

$$1 = \beta\big(1 + r_t(x, \digamma)\big) \sum_{\xi \in \mathcal{X}, \sigma \in \mathcal{S}} U'\Big(v^{\xi,\sigma}_{t,x,s}(c, \digamma)/c\Big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma),$$

$$1 = \beta\alpha\big(K_t(x, \digamma)/N\big)^{\alpha-1} \sum_{\xi \in \mathcal{X}, \sigma \in \mathcal{S}} U'\Big(v^{\xi,\sigma}_{t,x,s}(c, \digamma)/c\Big)\xi\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma),$$

$$\theta_{t,x,s}(c, \digamma)(1 + r_t(x, \digamma)) + \vartheta_{t,x,s}(c, \digamma)\left(\xi\alpha\frac{K_t(x, \digamma)^{\alpha-1}}{N^{\alpha-1}} - \delta\right) + \xi(1-\alpha)\frac{K_t(x, \digamma)^{\alpha}}{N^{\alpha}}\sigma$$

$$= v^{\xi,\sigma}_{t,x,s}(c, \digamma) + \theta_{t+1,\xi,\sigma}\Big(v^{\xi,\sigma}_{t,x,s}(c, \digamma), \tilde{\digamma}^{\xi}\Big) + \vartheta_{t+1,\xi,\sigma}\Big(v^{\xi,\sigma}_{t,x,s}(c, \digamma), \tilde{\digamma}^{\xi}\Big), \quad \xi \in \mathcal{X}, \ \sigma \in \mathcal{S}.$$

$\circledast$

The Dual (Principle of Maximum) Approach to GEI

At time $t = T$ there is no future $\Rightarrow$ $\theta_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0,\quad \vartheta_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0,\quad v^{\xi,\sigma}_{T,x,s}(c,\digamma) \stackrel{\text{def}}{=} 0.$

Suppose that for some fixed $0 \le t < T$ the following demand functions are known

$$\mathbb{R}_{++} \times \mathbb{F}^{\mathcal{S}} \ni (c,\digamma) \quad \rightsquigarrow \quad \theta_{t+1,\xi,\sigma}(c,\digamma) \in \mathbb{R}, \quad \vartheta_{t+1,\xi,\sigma}(c,\digamma), \quad \text{for all} \quad \xi \in \mathcal{X} \quad \text{and} \quad \sigma \in \mathcal{S}.$$

Given $x \in \mathcal{X}$, choose and fix $\digamma \in \mathbb{F}$, and make an ansatz choice for $r_t(x,\digamma) \in \mathbb{R}$ and $K_t(x,\digamma) \in \mathbb{R}_+$.

For every fixed $(s,c) \in \mathcal{S} \times \mathbb{R}_{++}$, consider the following system of $2 + |\mathcal{X}| \times |\mathcal{S}|$ equations, parameterized by $\left(\tilde{\digamma}^{\xi} \in \mathbb{F}^{\mathcal{S}},\ \xi \in \mathcal{X}\right)$, for the (same exact number of) unknowns $\theta_{t,x,s}(c,\digamma)$, $\vartheta_{t,x,s}(c,\digamma)$, and $v^{\xi,\sigma}_{t,x,s}(c,\digamma)$, $\xi \in \mathcal{X}$, $\sigma \in \mathcal{S}$:

$$
\begin{aligned}
1 &= \beta\big(1 + r_t(x,\digamma)\big) \sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} U'\Big(v^{\xi,\sigma}_{t,x,s}(c,\digamma)/c\Big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma),\\
1 &= \beta\alpha\Big(K_t(x,\digamma)/N\Big)^{\alpha-1} \sum\nolimits_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} U'\Big(v^{\xi,\sigma}_{t,x,s}(c,\digamma)/c\Big)\xi\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma),\\
&\theta_{t,x,s}(c,\digamma)(1 + r_t(x,\digamma)) + \vartheta_{t,x,s}(c,\digamma)\Big(\xi\alpha\frac{K_t(x,\digamma)^{\alpha-1}}{N^{\alpha-1}} - \delta\Big) + \xi(1-\alpha)\frac{K_t(x,\digamma)^{\alpha}}{N^{\alpha}}\sigma\\
&= v^{\xi,\sigma}_{t,x,s}(c,\digamma) + \theta_{t+1,\xi,\sigma}\Big(v^{\xi,\sigma}_{t,x,s}(c,\digamma),\tilde{\digamma}^{\xi}\Big) + \vartheta_{t+1,\xi,\sigma}\Big(v^{\xi,\sigma}_{t,x,s}(c,\digamma),\tilde{\digamma}^{\xi}\Big), \quad \xi\in\mathcal{X},\ \sigma\in\mathcal{S}.
\end{aligned}
$$

*

Solving the system for all the choices of $(s,c) \in \mathcal{S} \times \mathbb{R}_{++}$ gives the functions:

$$(s,c) \rightsquigarrow \theta_{t,x,s}(c,\digamma), \quad (s,c) \rightsquigarrow \vartheta_{t,x,s}(c,\digamma), \quad \text{and} \quad (s,c) \rightsquigarrow v^{\xi,\sigma}_{t,x,s}(c,\digamma), \quad \xi \in \mathcal{X}, \quad \sigma \in \mathcal{S}.$$

The Dual (Principle of Maximum) Approach to GEI

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Now test the identities (between functions on $\mathbb{R}_{++}$)

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If at least one of the identities in $\star$ fails, the choice of $\tilde{F}^{\xi}\in\mathbb{F}^{\mathcal{S}}$, $\xi\in\mathcal{X}$ is to be modified accordingly (e.g., the right side of $\star$ could be the next choice) and $\circledast$ is to be solved again – until $\star$ is satisfied.

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gets accepted temporarily, and the following two market clearing conditions are to be tested:

$$\sum_{s\in\mathcal{S}}\pi(s)\int_0^\infty \theta_{t,x,s}(c,F)\,\mathrm{d}F_s(c) \overset{?}{=} 0, \qquad \sum_{s\in\mathcal{S}}\pi(s)\int_0^\infty \vartheta_{t,x,s}(c,F)\,\mathrm{d}F_s(c) \overset{?}{=} K_t(x,F). \qquad ☼$$

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Finally, the same procedure is repeated with all other choices for $x\in\mathcal{X}$.

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After that the recursion can proceed to period $(t-1)$ – all the way to period $t=0$.

The Dual (Principle of Maximum) Approach to GEI

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Since all agents are exactly identical before the time $t = 0$ aggregate shock $x \in \mathcal{X}$ and all private shocks $s_i \in \mathcal{S}$, $i \in \mathcal{H}$, are realized, the actual (physical) distribution of the households, $F^{\text{ini}} \in \mathbb{F}^{\mathcal{S}}$, must be such that ${F^{\text{ini}}}_s = \int \delta_{c_s}$, for every $s \in \mathcal{S}$.

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$$\theta_{0,x,c_s}(c_s, F^{\text{ini}}) + \vartheta_{0,x,s}(c_s, F^{\text{ini}}) + c_s = x(1-\alpha)\frac{{K_{-1}}^{\alpha}}{N^{\alpha}}s\,, \quad s \in \mathcal{S}\,.$$

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All initial portfolios $\theta_{0,x,s}(c_s, F^{\mathrm{ini}})$ and $\vartheta_{0,x,s}(c_s, F^{\mathrm{ini}})$, $s \in \mathcal{S}$, are now fully determined, and so are the period $t = 1$ private consumption plans $v^{\xi,\sigma}_{t,x,s}(c_s, F^{\mathrm{ini}})$, $x \in \mathcal{X}$, $s, \sigma \in \mathcal{S}$, and collective consumption choice $\tilde{F}^{\xi} \in \mathbb{F}^{\mathcal{S}}$, $\xi \in \mathcal{X}$, from

$$\tilde{F}^{\xi}{}_{\sigma}(c) = \sum_{s\in\mathcal{S}} \frac{\pi(s)\mathcal{P}_{s,\sigma}}{\pi(\sigma)}(\mathrm{d}F^{\mathrm{ini}}_s)\Big(v^{\xi,\sigma}_{0,x,s}(\cdot, F^{\mathrm{ini}})^{-1}(]0,c])\Big)\,, \quad \text{for all } c \in \mathbb{R}_{++}, \quad \sigma \in \mathcal{S} \quad \xi \in \mathcal{X}.$$

Bewley-Aiyagari-Huggett Models

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With $U(c)=(c^{1-R}-1)/R$, the FOCs for household "c" in state $(x,s)\in\mathcal{X}\times\mathcal{S}$ are:

$$1=\beta\sum\nolimits_{\xi\in\mathcal{X},\,\sigma\in\mathcal{S}}\left(\frac{c}{v_{x,s}^{\xi,\sigma}(c)}\right)^R\big(1+r(x)\big)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,,$$

$$1=\beta\sum\nolimits_{\xi\in\mathcal{X},\,\sigma\in\mathcal{S}}\left(\frac{c}{v_{x,s}^{\xi,\sigma}(c)}\right)^R\left(\xi\alpha\frac{K(x)^{\alpha-1}}{N^{\alpha-1}}-\delta\right)\mathcal{Q}_x(\xi)\mathcal{P}_s(\sigma)\,,$$

$$\theta_{x,s}(c)(1+r(x))+\vartheta_{x,s}(c)\left(\xi\alpha\frac{K(x)^{\alpha-1}}{N^{\alpha-1}}-\delta\right)+\xi(1-\alpha)\frac{K(x)^{\alpha}}{N^{\alpha}}\sigma$$
$$=v_{x,s}^{\xi,\sigma}(c)+\theta_{\xi,\sigma}(v_{x,s}^{\xi,\sigma}(c))+\vartheta_{\xi,\sigma}(v_{x,s}^{\xi,\sigma}(c))\,,\quad \xi\in\mathcal{X}\,,\ \sigma\in\mathcal{S}\,.$$

N.B. Consumption is both a state variable and a "dual" variable. The connection between the next period consumption and the present period consumption is essentially a connection between the next period covariables and the present period covariables – whence the parallel with the principle of maximum.

N.B. One is faced with infinitely many optimization problems that must be solved "in an orchestra," together with the market clearing

$$\sum\nolimits_{s\in\mathcal{S}}\pi_s\int_0^\infty\theta_{x,s}(c)\,\mathrm{d}F^x{}_s(c)=0\quad\text{and}\quad\sum\nolimits_{s\in\mathcal{S}}\pi_s\int_0^\infty\vartheta_{x,s}(c)\,\mathrm{d}F^x{}_s(c)=K(x)\,,\quad x\in\mathcal{X}\,.$$

Bewley-Aiyagari-Huggett Models

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(c) This coordination is also a connection between the Lagrange multipliers attached to two consecutive time periods – principle of maximum again.

Bewley-Aiyagari-Huggett Models

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Solving for the equilibrium comes down to computing the following functions (of private consumption)

$$\theta_{x,s}(\cdot),\ \vartheta_{x,s}(\cdot),\ \nu_{x,s}^{\xi,\sigma}(\cdot),\ \text{and}\ F^{x}{}_{s}(\cdot),\quad \text{for } x,\xi \in \mathcal{X},\ s,\sigma \in \mathcal{S}\,.$$

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Market clearing: $$\sum_{s\in\mathcal{S}}\pi_s\int_0^\infty\theta_{x,s}(c)\,\mathrm{d}F^{x}{}_{s}(c)=0\quad\text{and}\quad\sum_{s\in\mathcal{S}}\pi_s\int_0^\infty\vartheta_{x,s}(c)\,\mathrm{d}F^{x}{}_{s}(c)=K(x)\,.$$

Bewley-Aiyagari-Huggett Models

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N.B. There is an *endogenous* upper bound on consumption given by:

$$\bar{c} = \inf\{c \in \mathbb{R}_{++} : \, v_{x,s}^{\xi,\sigma}(c) \le c \text{ for every } x, \xi \in \mathcal{X} \text{ and } s, \sigma \in \mathcal{S}\} .$$

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Step 0: Make an ansatz choice for $K(x) \in \mathbb{R}_+$ and $r(x) \in \mathbb{R}$, $x \in \mathcal{X}$ (a total of $2|\mathcal{X}|$ scalar values). Make an ansatz choice for the collection of portfolio functions $\tilde{\theta}_{x,s}(\cdot)$, $\tilde{\vartheta}_{x,s}(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$. Go to step 1.

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$$\theta_{x,s}(c)(1 + r(x)) + \vartheta_{x,s}(c)\left(1 + \xi\alpha\frac{K(x)^{\alpha-1}}{N^{\alpha-1}} - \delta\right) + \xi(1-\alpha)\frac{K(x)^{\alpha}}{N^{\alpha}}\sigma = v_{x,s}^{\xi,\sigma}(c) + \tilde{\theta}_{\xi,\sigma}(v_{x,s}^{\xi,\sigma}(c)) + \tilde{\vartheta}_{\xi,\sigma}(v_{x,s}^{\xi,\sigma}(c)),$$
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Step 2: For every choice of $x \in \mathcal{X}$, $s \in \mathcal{S}$, and $c \in \mathbb{G}_{]0,\bar{c}\,]}$, solve

$$\theta_{x,s}(c)\,(1+r(x)) + \vartheta_{x,s}(c)\Big(1+\xi\alpha\frac{K(x)^{\alpha-1}}{N^{\alpha-1}}-\delta\Big) + \xi(1-\alpha)\frac{K(x)^{\alpha}}{N^{\alpha}}\sigma = \nu^{\xi,\sigma}_{x,s}(c) + \tilde{\theta}_{\xi,\sigma}(\nu^{\xi,\sigma}_{x,s}(c)) + \tilde{\vartheta}_{\xi,\sigma}(\nu^{\xi,\sigma}_{x,s}(c)),\quad \xi\in\mathcal{X},\ \sigma\in\mathcal{S};$$

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By interpolating the respective values, define the functions $\theta_{x,s}(\cdot)$, $\vartheta_{x,s}(\cdot)$, and $\nu^{\xi,\sigma}_{x,s}(\cdot)$ as cubic splines over the grid $\mathbb{G}_{]0,\bar{c}\,]}$.

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for the unknowns $\theta_{x,s}(c)$, $\vartheta_{x,s}(c)$, and $\nu_{x,s}^{\xi,\sigma}(c)$, $\xi \in \mathcal{X}$, $s \in \mathcal{S}$.

By interpolating the respective values, define the functions $\theta_{x,s}(\cdot)$, $\vartheta_{x,s}(\cdot)$, and $\nu_{x,s}^{\xi,\sigma}(\cdot)$ as cubic splines over the grid $\mathbb{G}_{]0,\bar{c}\,]}$.

Define uniform interpolation grids on the ranges of the functions $\nu_{x,s}^{\xi,\sigma}(\cdot)$, compute the inverse values of those grid-points and, finally, define the inverse functions $(\nu_{x,s}^{\xi,\sigma})^{-1}(\cdot)$ as the cubic splines obtained by interpolating the inverse values over the respective grids.

Step 2: For every choice of $x \in \mathcal{X}$, $s \in \mathcal{S}$, and $c \in \mathbb{G}_{]0,\bar{c}\,]}$, solve

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Step 3: If this is the first visit to step 3, make an ansatz choice for the collection of distribution functions $\tilde{F}^x \in \mathbb{F}^S$, $x \in \mathcal{X}$ (say, choose these functions to give the uniform distribution on $]0, \bar{c}\,]$).

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Step 3: If this is the first visit to step 3, make an ansatz choice for the collection of distribution functions $\tilde{F}^x \in \mathbb{F}^{\mathcal{S}}$, $x \in \mathcal{X}$ (say, choose these functions to give the uniform distribution on $]0, \bar{c}\,]$). Otherwise, set $\tilde{F}^x$, $x \in \mathcal{X}$, to be the *most recently computed* versions of the distribution functions F^x. Go to step 4.

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Step 4: For every $\xi \in \mathcal{X}$, $\sigma \in \mathcal{S}$, and $c \in \mathbb{G}_{]0,\bar{c}\,]}$ compute

$$F^{\xi}{}_{\sigma}(c) = \sum_{x\in\mathcal{X},\, s\in\mathcal{S}} \frac{\psi_x \mathcal{Q}_x(\xi)}{\psi(\xi)} \frac{\pi_s \mathcal{P}_s(\sigma)}{\pi(\sigma)} \tilde{F}^x{}_s\big((\nu^{\xi,\sigma}_{x,s})^{-1}(c)\big) .$$

and produce an updated version of the distributions $\tilde{F}^x{}_s(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$, as cubic splines over the grid $\mathbb{G}_{]0,\bar{c}\,]}$ in the obvious way.

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and produce an updated version of the distributions $\tilde{F}^x{}_s(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$, as cubic splines over the grid $\mathbb{G}_{]0,\bar{c}\,]}$ in the obvious way.

Compute the error term

$$\max\nolimits_{\xi\in\mathcal{X},\,\sigma\in\mathcal{S},\,c\in\mathbb{G}_{]0,\bar{c}\,]}} \left|F^{\xi}{}_{\sigma}(c) - \tilde{F}^{\xi}{}_{\sigma}(c)\right|\,.$$

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and produce an updated version of the distributions $\tilde{F}^x{}_s(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$, as cubic splines over the grid $\mathbb{G}_{]0,\bar{c}\,]}$ in the obvious way.

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If this error term exceeds the prescribed threshold, set $\tilde{F}^x = F^x$, $x \in \mathcal{X}$, $s \in \mathcal{S}$, go back to the begging of the step and repeat.

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If this error term exceeds the prescribed threshold, set $\tilde{F}^x = F^x$, $x \in \mathcal{X}$, $s \in \mathcal{S}$, go back to the begging of the step and repeat. Otherwise record the most recently updated version of the distribution functions F^x, $x \in \mathcal{X}$, and proceed to the step 5.

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Step 5: Test the market clearing conditions with the latest $\mathcal{F}^x$, $x \in \mathcal{X}$,

$$\sum_{s\in\mathcal{S}} \pi_s \int_0^{\bar{c}} \theta_{x,s}(c)\,\mathrm{d}\mathcal{F}^x{}_s(c) = 0 \quad \text{and} \quad \sum_{s\in\mathcal{S}} \pi_s \int_0^{\bar{c}} \vartheta_{x,s}(c)\,\mathrm{d}\mathcal{F}^x{}_s(c) = K(x)\,,$$

in every aggregate state $x \in \mathcal{X}$, with the most recently updated versions of the functions $\theta_{x,s}(\cdot)$, $\vartheta_{x,s}(\cdot)$, and $\mathcal{F}^x{}_s(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$.

Step 5: Test the market clearing conditions with the latest $\mathcal{F}^x$, $x \in \mathcal{X}$,

$$\sum_{s\in\mathcal{S}} \pi_s \int_0^{\bar{c}} \theta_{x,s}(c)\,\mathrm{d}\mathcal{F}^x{}_s(c) = 0 \quad \text{and} \quad \sum_{s\in\mathcal{S}} \pi_s \int_0^{\bar{c}} \vartheta_{x,s}(c)\,\mathrm{d}\mathcal{F}^x{}_s(c) = K(x),$$

in every aggregate state $x \in \mathcal{X}$, with the most recently updated versions of the functions $\theta_{x,s}(\cdot)$, $\vartheta_{x,s}(\cdot)$, and $\mathcal{F}^x{}_s(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$.

If at least one of these identities fails by more than the prescribed threshold, in at least one aggregate state $x \in \mathcal{X}$, discard the spline objects $\theta_{x,s}(\cdot)$, $\vartheta_{x,s}(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$ (still having $\tilde{\theta}_{x,s}(\cdot)$, $\tilde{\vartheta}_{x,s}(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$ on record), modify the most recent choices for $K(x)$ and $r(x)$, $x \in \mathcal{X}$, accordingly, and go back to step 1. Otherwise go to step 6.

Step 5: Test the market clearing conditions with the latest F^x, $x \in \mathcal{X}$,

$$\sum_{s\in\mathcal{S}} \pi_s \int_0^{\bar{c}} \theta_{x,s}(c)\,\mathrm{d}F^x{}_s(c) = 0 \quad \text{and} \quad \sum_{s\in\mathcal{S}} \pi_s \int_0^{\bar{c}} \vartheta_{x,s}(c)\,\mathrm{d}F^x{}_s(c) = K(x),$$

in every aggregate state $x \in \mathcal{X}$, with the most recently updated versions of the functions $\theta_{x,s}(\cdot)$, $\vartheta_{x,s}(\cdot)$, and $F^x{}_s(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$.

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Remark: The most recently updated versions of the portfolio functions and the associated next-period-consumption functions do not get accepted until the market can be cleared with those new versions by adjusting $K(x)$ and $r(x)$, $x \in \mathcal{X}$, accordingly.

Bewley-Aiyagari-Huggett Models

Bewley-Aiyagari-Huggett Models

Bewley-Aiyagari-Huggett Models

Step 6: If this is the first visit to step 6, set $\tilde{\theta}_{x,s}(\cdot) = \theta_{x,s}(\cdot)$, $\tilde{\vartheta}_{x,s}(\cdot) = \vartheta_{x,s}(\cdot)$, $\tilde{\nu}^{\xi,\sigma}_{x,s}(\cdot) = \nu^{\xi,\sigma}_{x,s}(\cdot)$ and go back to step 1.

Step 6: If this is the first visit to step 6, set $\tilde{\theta}_{x,s}(\cdot) = \theta_{x,s}(\cdot)$, $\tilde{\vartheta}_{x,s}(\cdot) = \vartheta_{x,s}(\cdot)$, $\tilde{\nu}^{\xi,\sigma}_{x,s}(\cdot) = \nu^{\xi,\sigma}_{x,s}(\cdot)$ and go back to step 1.

Otherwise, compute the error terms

$$\max_{x\in\mathcal{X},\, s\in\mathcal{S},\, c\in\mathbb{G}_{]0,\bar{c}\,]}\left|\theta_{x,s}(c) - \tilde{\theta}_{x,s}(c)\right|, \quad \max_{x\in\mathcal{X},\, s\in\mathcal{S},\, c\in\mathbb{G}_{]0,\bar{c}\,]}\left|\vartheta_{x,s}(c) - \tilde{\vartheta}_{x,s}(c)\right|,$$

$$\text{and} \qquad \max_{x,\xi\in\mathcal{X},\, s,\sigma\in\mathcal{S},\, c\in\mathbb{G}_{]0,\bar{c}\,]}\left|\nu^{\xi,\sigma}_{x,s}(c) - \tilde{\nu}^{\xi,\sigma}_{x,s}(c)\right|.$$

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$$\max_{x\in\mathcal{X},\, s\in\mathcal{S},\, c\in\mathbb{G}_{]0,\bar{c}\,]}} |\theta_{x,s}(c) - \tilde{\theta}_{x,s}(c)|\,, \quad \max_{x\in\mathcal{X},\, s\in\mathcal{S},\, c\in\mathbb{G}_{]0,\bar{c}\,]}} |\vartheta_{x,s}(c) - \tilde{\vartheta}_{x,s}(c)|\,,$$

$$\text{and} \qquad \max_{x,\xi\in\mathcal{X},\, s,\sigma\in\mathcal{S},\, c\in\mathbb{G}_{]0,\bar{c}\,]}} |\nu^{\xi,\sigma}_{x,s}(c) - \tilde{\nu}^{\xi,\sigma}_{x,s}(c)|\,.$$

If at least one of these terms exceeds the prescribed threshold, set $\tilde{\theta}_{x,s}(\cdot) = \theta_{x,s}(\cdot)$, $\tilde{\vartheta}_{x,s}(\cdot) = \vartheta_{x,s}(\cdot)$, $\tilde{\nu}^{\xi,\sigma}_{x,s}(\cdot) = \nu^{\xi,\sigma}_{x,s}(\cdot)$ and go back to step 1.

Bewley-Aiyagari-Huggett Models

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$$\text{and} \qquad \max_{x,\xi\in\mathcal{X},\, s,\sigma\in\mathcal{S},\, c\in\mathbb{G}_{]0,\bar{c}]}} |\nu^{\xi,\sigma}_{x,s}(c) - \tilde{\nu}^{\xi,\sigma}_{x,s}(c)|\,.$$

If at least one of these terms exceeds the prescribed threshold, set $\tilde{\theta}_{x,s}(\cdot) = \theta_{x,s}(\cdot)$, $\tilde{\vartheta}_{x,s}(\cdot) = \vartheta_{x,s}(\cdot)$, $\tilde{\nu}^{\xi,\sigma}_{x,s}(\cdot) = \nu^{\xi,\sigma}_{x,s}(\cdot)$ and go back to step 1.

Otherwise stop. Declare that the equilibrium is given by the most recent choice for $K(x)$ and $r(x)$, $x \in \mathcal{X}$, the portfolio functions (constructed as cubic splines) $\theta_{x,s}(\cdot)$ and $\vartheta_{x,s}(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$, the most recently computed next-period-consumption mappings (also constructed as cubic splines) $\nu^{\xi,\sigma}_{x,s}(\cdot)$, $x, \xi \in \mathcal{X}$, $s, \sigma \in \mathcal{S}$, and the most recently updated family of distribution functions $F_{x,s}(\cdot)$, $x \in \mathcal{X}$, $s \in \mathcal{S}$ (splines as well).

Bewley-Aiyagari-Huggett Models

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The meta-program was implemented in the Julia programming language in the special case of no aggregate risk and no productive physical capital (only private IOUs).

Bewley-Aiyagari-Huggett Models

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The output (from the Julia program) is illustrated next in the context of the classical example of Huggett economy from (L&S, Ch. 18) – the same example in which the classical DP approach fails – and also in the context of examples in which both the DP and the dual methods work.

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The illustrations in the next few slides show that the proposed recursive program *can* find the equilibrium in situation where the classical approach fails.

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In the 7-states example from (L&S, Ch. 18) the program achieves convergence of $\approx 9.04753562\text{e-}5$ with 235 iterations, for ≈ 4.5 hours on a single processor (i7-8650U, OS: Fedora 30).

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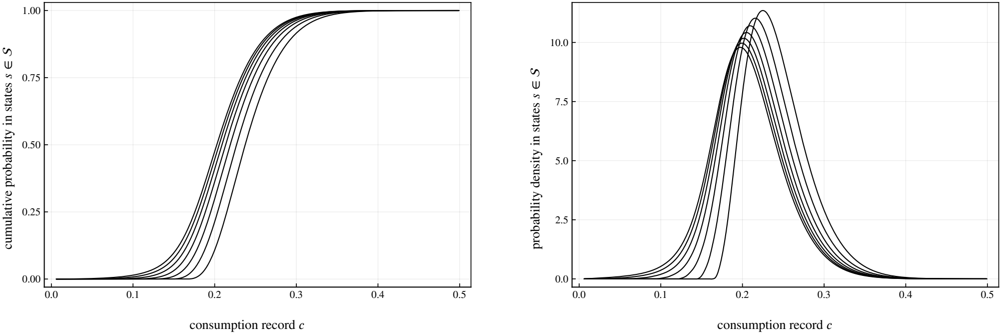
In the 7-states example from (L&S, Ch. 18) the program achieves convergence of ≈ 9.04753562e-5 with 235 iterations, for ≈ 4.5 hours on a single processor (i7-8650U, OS: Fedora 30).

The computed equilibrium rate is $r \approx 3.701851\%$, and the market is cleared with -1.76078736e-6. The endogenous upper bound on consumption is 0.91571955, which corresponds to asset holdings in the range $[-1.627487, 17.937506]$.

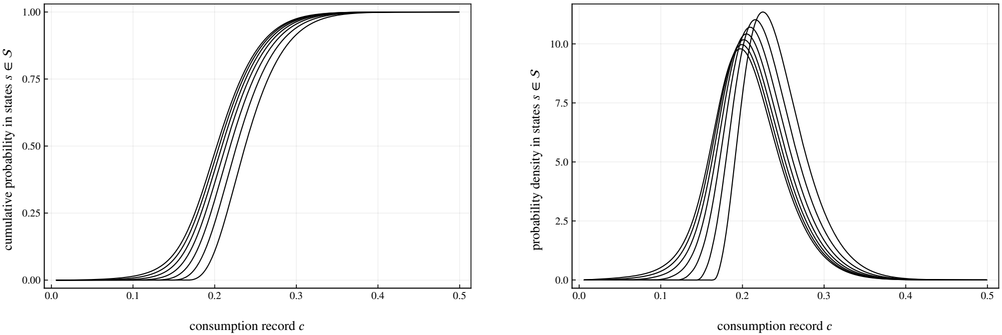
Bewley-Aiyagari-Huggett Models



The distribution of households in states $s \in \mathcal{S}$ over the range of consumption, obtained with the proposed dual approach.

N.B. The actual distribution of agents is amassed over a substantially smaller range than the endogenous domain.

Bewley-Aiyagari-Huggett Models



The distribution of households in states $s \in S$ over the range of consumption, obtained with the proposed dual approach.

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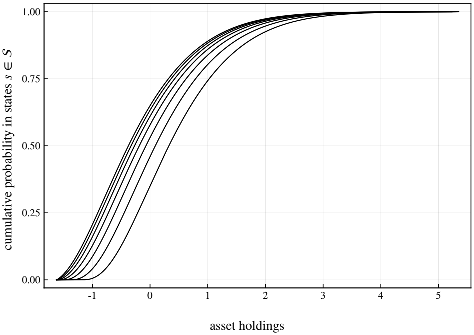
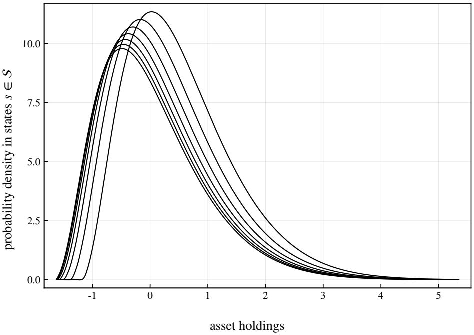
Bewley-Aiyagari-Huggett Models

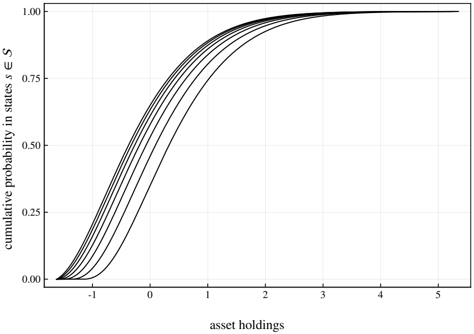


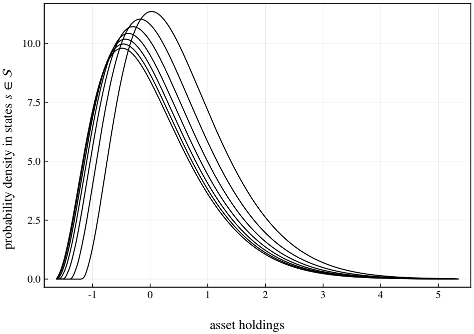


The distribution of households in states $s \in \mathcal{S}$ over the range of asset holdings, obtained with the proposed dual approach.

N.B. The actual distribution of agents is again amassed over a substantially smaller range than the endogenous domain.

Bewley-Aiyagari-Huggett Models





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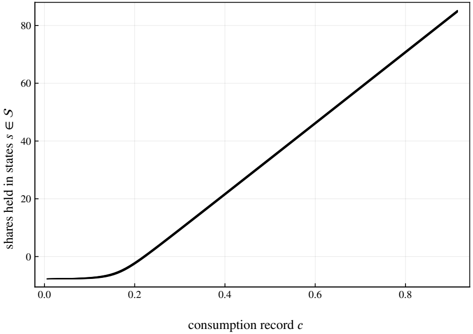
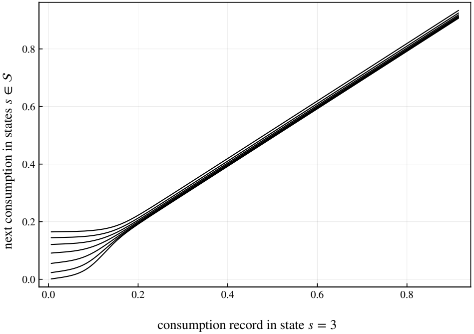
Bewley-Aiyagari-Huggett Models

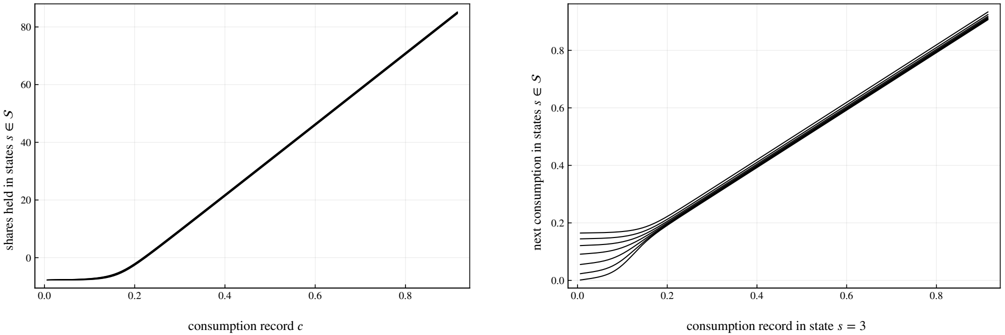




Left panel: the asset holdings (portfolios) as functions of consumption in all 7 classes of employment.

Right panel: the transfer of consumption from employment state 3 to all 7 employment states.

Bewley-Aiyagari-Huggett Models



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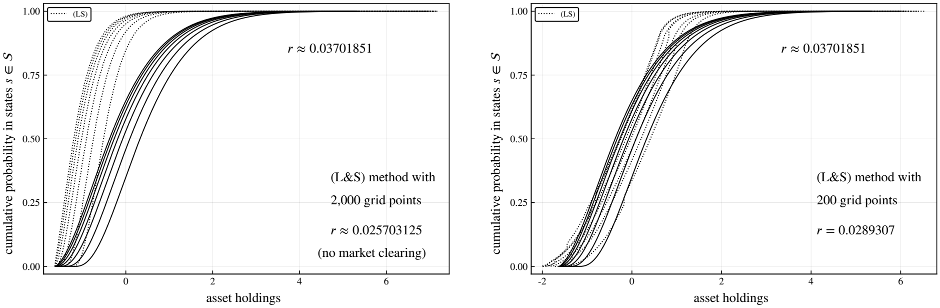
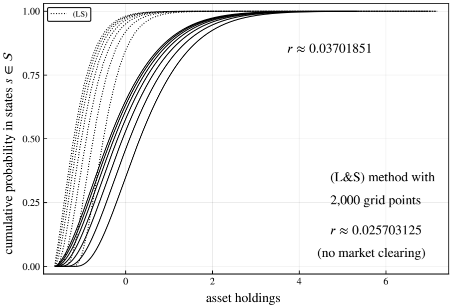
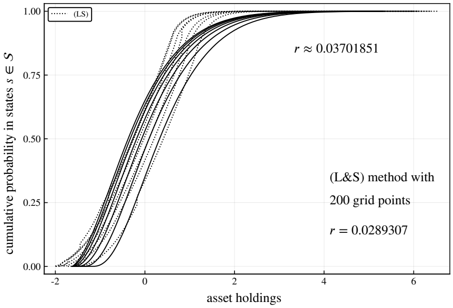


The output from the proposed new approach (the solid lines) shown against the output from the conventional dynamic programming approach in the 7-states example.

Bewley-Aiyagari-Huggett Models





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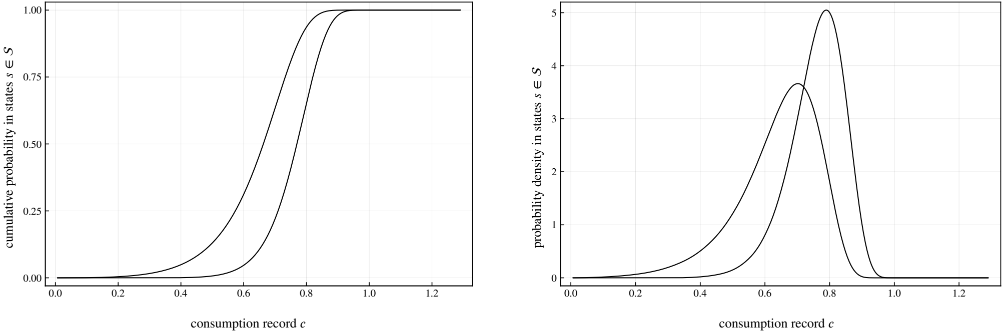
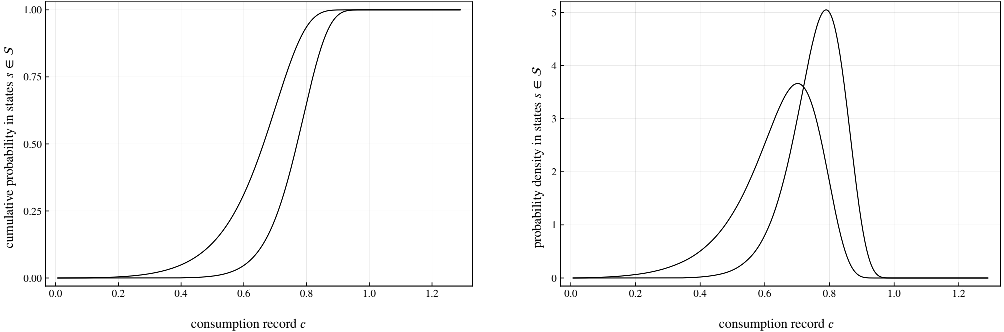


The distribution of households over the consumption range in the 2-states example.

Bewley-Aiyagari-Huggett Models



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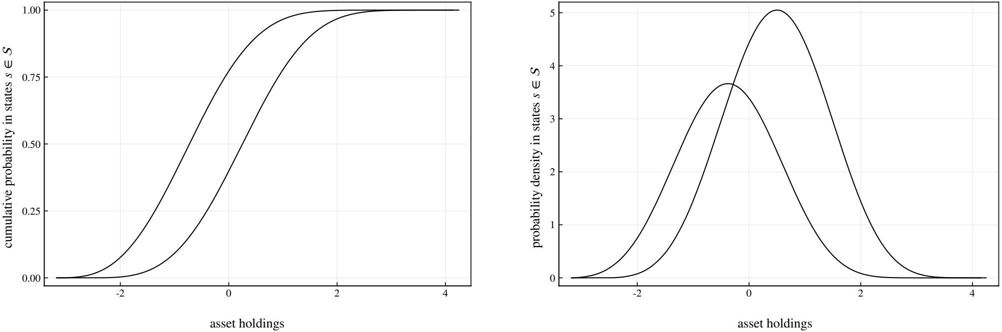
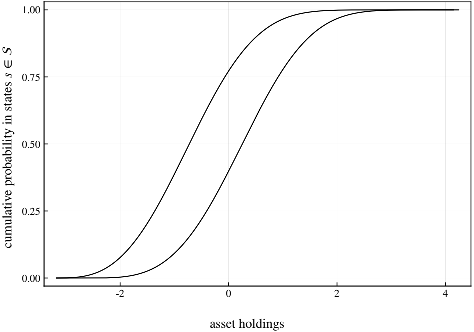
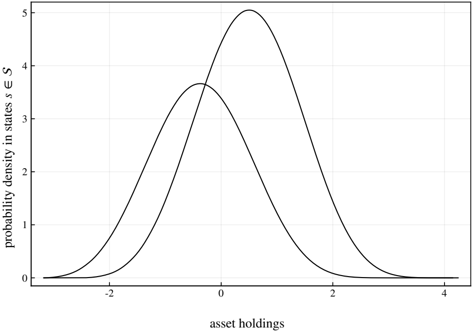


The distribution of households over wealth domain in the 2-states example.

Bewley-Aiyagari-Huggett Models





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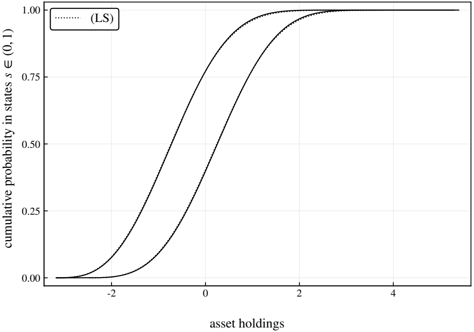
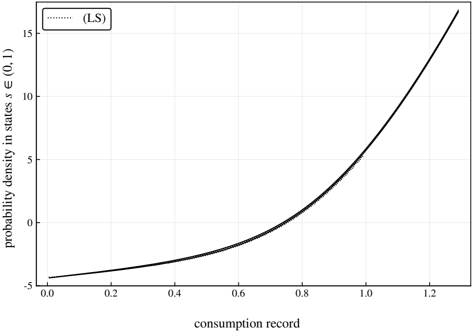
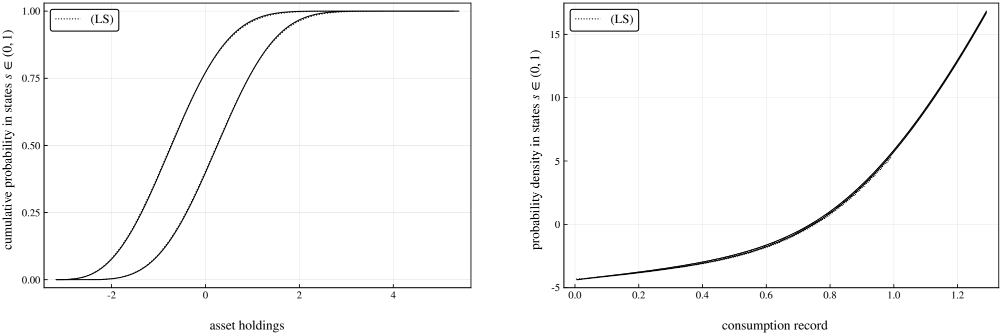




Comparison with the dynamic programming approach from (L&S) in the 2-states example.

Bewley-Aiyagari-Huggett Models



Comparison with the dynamic programming approach from (L&S) in the 2-states example.

Finite Mean Field Games and Control with Markovian Shocks

Finite Mean Field Games and Control with Markovian Shocks

Finite Mean Field Games and Control with Markovian Shocks

The private (idiosyncratic) state of any one particle follows a Markov chain with a (common for all particles) finite state space $\mathcal{S}$ and a (common for all particles) transition probability matrix $\mathcal{P}_s(\sigma)$, $s, \sigma \in \mathcal{S}$, which admits a unique set of steady-state probabilities $0 < \pi(s) < 1$, $s \in \mathcal{S}$.

Finite Mean Field Games and Control with Markovian Shocks

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All private Markov chains are independent from one another and are independent from everything else.

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The aggregate (shared by all particles) state, also follows a MC with a finite state space $\mathcal{X}$ and a transition probability matrix $\mathcal{Q}_x(\xi)$, $x, \xi \in \mathcal{X}$, with unique steady-state probabilities $0 < \psi(x) < 1$, $x \in \mathcal{X}$.

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All Markov chains are in steady state.

The position of every particle fluctuates in $\mathbb{R}$ and, letting $\mathscr{P}$ denote the family of all probability measures on $\mathbb{R}$, the collective distribution of all particles in period $t \in \mathbb{N}_T$, and in a given aggregate state $x \in \mathcal{X}$, can be described as an element $M \in \mathscr{P}^{\mathcal{S}}$, such that $\pi(s)M_s(A)$ gives the total mass of particles that are in (private) state s and are located at some position $z \in A$. The total mass of all particles is

$$\sum_{s\in\mathcal{S}} \pi(s)M_s(\mathbb{R}) = 1.$$

Finite Mean Field Games and Control with Markovian Shocks

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N.B. The collective distribution $M \in \mathscr{P}^{\mathcal{S}}$ depends on the aggregate state x and we may write M^x.

Finite Mean Field Games and Control with Markovian Shocks

Finite Mean Field Games and Control with Markovian Shocks

Finite Mean Field Games and Control with Markovian Shocks

Given a collective distribution $M \in \mathscr{P}^{\mathcal{S}}$ and an aggregate state $x \in \mathcal{X}$ in period $t \in \mathbb{N}_T$, the Bellman equation for a particle in idiosyncratic state $s \in \mathcal{S}$ that happens to be in location $z \in \mathbb{R}$ is:

$$V_{t,x,s}(z, M) = \max_{\alpha,\, Z(\xi,\sigma),\, \xi\in\mathcal{X},\sigma\in\mathcal{S}} \Big(f_{t,x,s}(\alpha, z, M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\, V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma), \mathscr{M}^{\xi}\big) \mathcal{P}_s(\sigma) \mathcal{Q}_x(\xi) \Big)$$

subject to:

$$Z(\xi,\sigma) - g^{\xi,\sigma}_{t,x,s}(\alpha, z, M) = z\,, \quad \xi \in \mathcal{X}, \sigma \in \mathcal{S},$$

where the family $(\mathscr{M}^{\xi} \in \mathscr{P}^{\mathcal{S}},\ \xi \in \mathcal{X})$, is treated as a parameter.

Finite Mean Field Games and Control with Markovian Shocks

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subject to:

$$Z(\xi,\sigma) - g^{\xi,\sigma}_{t,x,s}(\alpha, z, M) = z\,, \quad \xi \in \mathcal{X}, \sigma \in \mathcal{S},$$

where the family $(\mathscr{M}^{\xi} \in \mathscr{P}^{\mathcal{S}},\ \xi \in \mathcal{X})$, is treated as a parameter. The Lagrangian is:

$$\begin{aligned}\mathcal{L}\big(\alpha, Z(\xi,\sigma), \Phi_{t+1,\xi,\sigma}, \xi \in \mathcal{X}, \sigma \in \mathcal{S}\big) &= f_{t,x,s}(\alpha, z, M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\, V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma), \mathscr{M}^{\xi}\big)\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi) \\ &\quad + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho \times \Big(z - Z(\xi,\sigma) + g^{\xi,\sigma}_{t,x,s}(\alpha, z, M)\Big)\Phi_{t+1,\xi,\sigma}\, \mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)\,,\end{aligned}$$

Finite Mean Field Games and Control with Markovian Shocks

Given a collective distribution $M \in \mathscr{P}^{\mathcal{S}}$ and an aggregate state $x \in \mathcal{X}$ in period $t \in \mathbb{N}_T$, the Bellman equation for a particle in idiosyncratic state $s \in \mathcal{S}$ that happens to be in location $z \in \mathbb{R}$ is:

$$V_{t,x,s}(z,M) = \max_{\alpha,\, Z(\xi,\sigma),\, \xi\in\mathcal{X},\sigma\in\mathcal{S}} \Big(f_{t,x,s}(\alpha,z,M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\, V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma),\mathscr{M}^{\xi}\big)\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)\Big)$$

subject to:

$$Z(\xi,\sigma) - g^{\xi,\sigma}_{t,x,s}(\alpha,z,M) = z\,, \quad \xi\in\mathcal{X},\sigma\in\mathcal{S},$$

where the family $(\mathscr{M}^{\xi}\in\mathscr{P}^{\mathcal{S}},\ \xi\in\mathcal{X})$, is treated as a parameter. The Lagrangian is:

$$\mathcal{L}\big(\alpha, Z(\xi,\sigma), \Phi_{t+1,\xi,\sigma}, \xi\in\mathcal{X},\sigma\in\mathcal{S}\big) = f_{t,x,s}(\alpha,z,M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\, V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma),\mathscr{M}^{\xi}\big)\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)$$
$$+ \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\times\Big(z - Z(\xi,\sigma) + g^{\xi,\sigma}_{t,x,s}(\alpha,z,M)\Big)\Phi_{t+1,\xi,\sigma}\,\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)\,,$$

and the FOCs are ($\nabla \overset{\text{def}}{=} \partial_{\text{space}}$, $\mathcal{D} \overset{\text{def}}{=} \partial_{\text{control}}$):

$$Z(\xi,\sigma) = z + g^{\xi,\sigma}_{t,x,s}(\alpha,z,M)\,, \quad \Phi_{t+1,\xi,\sigma} = \nabla V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma),\mathscr{M}^{\xi}\big)\,, \ \text{for every}\ \xi\in\mathcal{X}\ \text{and}\ \sigma\in\mathcal{S}\,,$$
$$\text{and}\quad \mathcal{D}f_{t,x,s}(\alpha,z,M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\times\mathcal{D}g^{\xi,\sigma}_{t,x,s}(\alpha,z,M)\Phi_{t+1,\xi,\sigma}\,\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi) = 0\,.$$

Finite Mean Field Games and Control with Markovian Shocks

Given a collective distribution $M \in \mathscr{P}^{\mathcal{S}}$ and an aggregate state $x \in \mathcal{X}$ in period $t \in \mathbb{N}_T$, the Bellman equation for a particle in idiosyncratic state $s \in \mathcal{S}$ that happens to be in location $z \in \mathbb{R}$ is:

$$V_{t,x,s}(z,M) = \max_{\alpha,\, Z(\xi,\sigma),\, \xi\in\mathcal{X},\sigma\in\mathcal{S}} \Big(f_{t,x,s}(\alpha,z,M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\, V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma),\mathscr{M}^{\xi}\big)\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)\Big)$$

subject to:

$$Z(\xi,\sigma) - g^{\xi,\sigma}_{t,x,s}(\alpha,z,M) = z\,, \quad \xi\in\mathcal{X},\sigma\in\mathcal{S},$$

where the family $(\mathscr{M}^{\xi} \in \mathscr{P}^{\mathcal{S}},\ \xi \in \mathcal{X})$, is treated as a parameter. The Lagrangian is:

$$\mathcal{L}\big(\alpha, Z(\xi,\sigma), \Phi_{t+1,\xi,\sigma}, \xi\in\mathcal{X},\sigma\in\mathcal{S}\big) = f_{t,x,s}(\alpha,z,M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\, V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma),\mathscr{M}^{\xi}\big)\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)$$
$$+ \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\times\Big(z - Z(\xi,\sigma) + g^{\xi,\sigma}_{t,x,s}(\alpha,z,M)\Big)\Phi_{t+1,\xi,\sigma}\,\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)\,,$$

and the FOCs are ($\nabla \stackrel{\text{def}}{\equiv} \partial_{\text{space}}$, $\mathcal{D} \stackrel{\text{def}}{\equiv} \partial_{\text{control}}$):

$$Z(\xi,\sigma) = z + g^{\xi,\sigma}_{t,x,s}(\alpha,z,M)\,, \quad \Phi_{t+1,\xi,\sigma} = \nabla V_{t+1,\xi,\sigma}\big(Z(\xi,\sigma),\mathscr{M}^{\xi}\big)\,, \text{ for every } \xi\in\mathcal{X} \text{ and } \sigma\in\mathcal{S}\,,$$
$$\text{and} \quad \mathcal{D}f_{t,x,s}(\alpha,z,M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\times\mathcal{D}g^{\xi,\sigma}_{t,x,s}(\alpha,z,M)\Phi_{t+1,\xi,\sigma}\,\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi) = 0\,.$$

By the envelope theorem:

$$\Phi_{t,x,s} \equiv \nabla V_{t,x,s}(z,M) = \nabla f_{t,x,s}(\alpha,z,M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho\times\Big(1 + \nabla g^{\xi,\sigma}_{t,x,s}(\alpha,z,M)\Big)\Phi_{t+1,\xi,\sigma}\,\mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)\,.$$

Finite Mean Field Games and Control with Markovian Shocks

Finite Mean Field Games and Control with Markovian Shocks

Finite Mean Field Games and Control with Markovian Shocks

Let $\Phi_{T+1,x,s}(z, M) \stackrel{\text{def}}{=} 0$ and let $\alpha_{T,x,s}(z, M)$ be the solution, α, of the equation

$$\mathcal{D}f_{T,x,s}(\alpha, z, M) = 0\,, \quad \text{for } x \in \mathcal{X},\ s \in \mathcal{S},\ z \in \mathbb{R}, \text{and } M \in \mathscr{P}^{\mathcal{S}}.$$

Finite Mean Field Games and Control with Markovian Shocks

Let $\Phi_{T+1,x,s}(z,M) \stackrel{\text{def}}{=} 0$ and let $\alpha_{T,x,s}(z,M)$ be the solution, α, of the equation

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Let $0 \leq t < T$ and suppose that the following functions are known (given, computed):

$$\mathbb{R} \times \mathscr{P}^{\mathcal{S}} \ni (z,M) \quad \rightsquigarrow \quad \alpha_{t+1,x,s}(z,M)\,,\ \Phi_{t+2,x,s}(z,M)\,, \quad \text{for all } x \in \mathcal{X} \text{ and } s \in \mathcal{S}\,.$$

Finite Mean Field Games and Control with Markovian Shocks

Let $\Phi_{T+1,x,s}(z,M) \overset{\text{def}}{=} 0$ and let $\alpha_{T,x,s}(z,M)$ be the solution, α, of the equation

$$\mathcal{D}f_{T,x,s}(\alpha,z,M)=0\,,\quad \text{for } x\in\mathcal{X},\ s\in\mathcal{S},\ z\in\mathbb{R}, \text{ and } M\in\mathscr{P}^{\mathcal{S}}.$$

Let $0\le t<T$ and suppose that the following functions are known (given, computed):

$$\mathbb{R}\times\mathscr{P}^{\mathcal{S}}\ni(z,M)\quad\rightsquigarrow\quad\alpha_{t+1,x,s}(z,M)\,,\ \Phi_{t+2,x,s}(z,M)\,,\quad \text{for all } x\in\mathcal{X} \text{ and } s\in\mathcal{S}\,.$$

Construct the mappings:

$$\mathbb{R}\ni z\ \rightsquigarrow\ \mathscr{T}^{\xi,\sigma}_{t+1,x,s,M}(z)\overset{\text{def}}{=}z+g^{\xi,\sigma}_{t+1,x,s}\Big(\alpha_{t+1,x,s}(z,M),z,M\Big)\,,\quad x,\xi\in\mathcal{X}\,,\ s,\sigma\in\mathcal{S}\,,\ M\in\mathscr{P}^{\mathcal{S}}\,,$$

Let $\Phi_{T+1,x,s}(z,M) \stackrel{\text{def}}{=} 0$ and let $\alpha_{T,x,s}(z,M)$ be the solution, α, of the equation

$$\mathcal{D}f_{T,x,s}(\alpha,z,M)=0\,,\quad \text{for } x\in\mathcal{X},\ s\in\mathcal{S},\ z\in\mathbb{R}, \text{ and } M\in\mathscr{P}^{\mathcal{S}}.$$

Let $0 \leq t < T$ and suppose that the following functions are known (given, computed):

$$\mathbb{R}\times\mathscr{P}^{\mathcal{S}} \ni (z,M) \quad \rightsquigarrow \quad \alpha_{t+1,x,s}(z,M)\,,\ \Phi_{t+2,x,s}(z,M)\,,\quad \text{for all } x\in\mathcal{X} \text{ and } s\in\mathcal{S}\,.$$

Construct the mappings:

$$\mathbb{R}\ni z \ \rightsquigarrow\ \mathscr{T}^{\xi,\sigma}_{t+1,x,s,M}(z) \stackrel{\text{def}}{=} z + g^{\xi,\sigma}_{t+1,x,s}\big(\alpha_{t+1,x,s}(z,M),z,M\big)\,,\quad x,\xi\in\mathcal{X}\,,\ s,\sigma\in\mathcal{S}\,,\ M\in\mathscr{P}^{\mathcal{S}}\,,$$

and then the mappings $\mathscr{P}^{\mathcal{S}}\ni M \rightsquigarrow \mathfrak{O}^{\xi}_{t+1,x}(M)\in\mathscr{P}^{\mathcal{S}},\ x,\xi\in\mathcal{X}$, so that

$$\mathfrak{O}^{\xi}_{t+1,x}(M)_{\sigma} \stackrel{\text{def}}{=} \sum_{s\in\mathcal{S}} \frac{\pi(s)\mathcal{P}_s(\sigma)}{\pi(\sigma)} \times M_s\circ\big(\mathscr{T}^{\xi,\sigma}_{t+1,x,s,M}\big)^{-1}\,,\quad \text{for every } \sigma\in\mathcal{S}\,.$$

Let $\boldsymbol{\Phi}_{T+1,x,s}(z,M) \overset{\text{def}}{=} 0$ and let $\alpha_{T,x,s}(z,M)$ be the solution, α, of the equation

$$\mathcal{D} f_{T,x,s}(\alpha, z, M) = 0\,, \quad \text{for } x \in \mathcal{X},\ s \in \mathcal{S},\ z \in \mathbb{R}, \text{ and } M \in \mathscr{P}^{\mathcal{S}}.$$

Let $0 \le t < T$ and suppose that the following functions are known (given, computed):

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Construct the mappings:

$$\mathbb{R} \ni z \ \rightsquigarrow\ \mathscr{T}^{\xi,\sigma}_{t+1,x,s,M}(z) \overset{\text{def}}{=} z + g^{\xi,\sigma}_{t+1,x,s}\big(\alpha_{t+1,x,s}(z,M), z, M\big)\,, \quad x, \xi \in \mathcal{X}\,,\ s, \sigma \in \mathcal{S}\,,\ M \in \mathscr{P}^{\mathcal{S}}\,,$$

and then the mappings $\mathscr{P}^{\mathcal{S}} \ni M \rightsquigarrow \Game^{\xi}_{t+1,x}(M) \in \mathscr{P}^{\mathcal{S}}$, $x, \xi \in \mathcal{X}$, so that

$$\Game^{\xi}_{t+1,x}(M)_\sigma \overset{\text{def}}{=} \sum_{s\in\mathcal{S}} \frac{\pi(s)\mathcal{P}_s(\sigma)}{\pi(\sigma)} \times M_s \circ \big(\mathscr{T}^{\xi,\sigma}_{t+1,x,s,M}\big)^{-1}\,, \quad \text{for every } \sigma \in \mathcal{S}\,.$$

Finally, for every $x \in \mathcal{X}$ and $s \in \mathcal{S}$, define the mappings

$$\begin{aligned}
\mathbb{R} \times \mathscr{P}^{\mathcal{S}} \ni (z, M) \quad \rightsquigarrow \quad & \boldsymbol{\Phi}_{t+1,x,s}(z,M) \\
= \nabla f_{t+1,x,s}\big(\alpha_{t+1,x,s}(z,M), z, M\big) & \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho \times \Big(1 + \nabla g^{\xi,\sigma}_{t+1,x,s}\big(\alpha_{t+1,x,s}(z,M), z, M\big)\Big) \\
& \times \boldsymbol{\Phi}_{t+2,\xi,\sigma}\big(\mathscr{T}^{\xi,\sigma}_{t+1,x,s,M}(z), \Game^{\xi}_{t+1,x}(M)\big)\, \mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi)\,.
\end{aligned}$$

Finite Mean Field Games and Control with Markovian Shocks

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For a fixed aggregate state $x \in \mathscr{X}$ and fixed cross-sectional distribution $M \in \mathscr{P}^{\mathcal{S}}$ (attached to x), consider a particle that happens to be in state $s \in \mathcal{S}$ and in location $z \in \mathbb{R}$.

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$$\mathcal{D}f_{t,x,s}(\alpha, z, M) + \sum_{\xi\in\mathcal{X},\sigma\in\mathcal{S}} \rho \times \mathcal{D}g^{\xi,\sigma}_{t,x,s}(\alpha, z, M)\, \Phi_{t+1,\xi,\sigma}\big(z + g^{\xi,\sigma}_{t,x,s}(\alpha, z, M), \mathscr{M}^{\xi}\big)\, \mathcal{P}_s(\sigma)\mathcal{Q}_x(\xi) = 0\,,$$

in which $\mathscr{M}^{\xi} \in \mathscr{P}^{\mathcal{S}}$, $\xi \in \mathcal{X}$, are treated as parameters.

By solving for various choices of $z \in \mathbb{R}$ and $s \in \mathcal{S}$, construct the mappings (x and M are fixed)

$$\mathbb{R} \ni z \ \rightsquigarrow\ \mathscr{T}^{\xi,\sigma}_{t,x,s,M}(z) \stackrel{\text{def}}{=} z + g^{\xi,\sigma}_{t,x,s}\big(\alpha_{t,x,s}(z, M), z, M\big)\,, \quad \xi \in \mathcal{X}\,,\ s, \sigma \in \mathcal{S}\,,$$

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and test the identities

$$\mathscr{M}^{\xi}{}_{\sigma} \stackrel{?}{=} \sum_{s\in\mathcal{S}} \frac{\pi(s)\mathcal{P}_s(\sigma)}{\pi(\sigma)} \times M_s \circ \left(\mathscr{T}^{\xi,\sigma}_{t,x,s,M}\right)^{-1}, \quad \text{for every } \xi \in \mathcal{X} \text{ and every } \sigma \in \mathcal{S}. \qquad ✓$$

Finite Mean Field Games and Control with Markovian Shocks

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If at least one of these identities fails, modify the family $\mathscr{M}^{\xi} \in \mathscr{P}^{\mathcal{S}}$, $\xi \in \mathcal{X}$, accordingly (e.g., use the right sides as the next guess) and repeat – until all identities in ✓ hold.

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After all relations in ✓ are attained, the solutions $z \rightsquigarrow \alpha_{t,x,s}(z, M)$, $s \in \mathcal{S}$, get accepted, and the entire procedure is repeated with other choices for $x \in \mathcal{X}$ and $M \in \mathscr{P}^{\mathcal{S}}$ – until the functions

$$\mathbb{R} \times \mathscr{P}^{\mathcal{S}} \ni (z, M) \;\rightsquigarrow\; \alpha_{t,x,s}(z, M)\,, \quad x \in \mathcal{X}\,,\ s \in \mathcal{S}\,,$$

become known.

For a fixed aggregate state $x \in \mathcal{X}$ and fixed cross-sectional distribution $M \in \mathscr{P}^{\mathcal{S}}$ (attached to x), consider a particle that happens to be in state $s \in \mathcal{S}$ and in location $z \in \mathbb{R}$. The particle computes its control $\alpha_{t,x,s}(z, M)$ by solving for α from the equation

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in which $\mathscr{M}^{\xi} \in \mathscr{P}^{\mathcal{S}}$, $\xi \in \mathcal{X}$, are treated as parameters.

By solving for various choices of $z \in \mathbb{R}$ and $s \in \mathcal{S}$, construct the mappings (x and M are fixed)

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After all relations in ✓ are attained, the solutions $z \rightsquigarrow \alpha_{t,x,s}(z, M)$, $s \in \mathcal{S}$, get accepted, and the entire procedure is repeated with other choices for $x \in \mathcal{X}$ and $M \in \mathscr{P}^{\mathcal{S}}$ – until the functions

$$\mathbb{R} \times \mathscr{P}^{\mathcal{S}} \ni (z, M) \;\rightsquigarrow\; \alpha_{t,x,s}(z, M)\,, \quad x \in \mathcal{X}\,,\ s \in \mathcal{S}\,,$$

become known. The recursion can now proceed to period $(t-1)$ – all the way to $t = 0$.

Decoupling Fields (an Illustration)

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Without mean field interactions, a generic stochastic optimal control problem can be stated as:

$$\begin{aligned} &\underset{\alpha_t,\ldots,\alpha_T}{\text{maximize}}\left(f_{t,\omega_t}(X_t,\alpha_t) + \mathsf{E}_t\left[\sum_{s=t+1}^{T} f_{s,\omega_s}(X_s,\alpha_s)\right]\right), \quad 0 \le t \le T\,, \\ &\text{subject to} \quad X_{s+1} = X_s + g_{s,\omega_{s+1}}(X_s,\alpha_s)\,, \quad s = t,\ldots,T\,, \end{aligned} \tag{*}$$

$\alpha_t \overset{\text{def}}{=}$ the choice of control, $\omega_t \overset{\text{def}}{=}$ the realized uncertain state (MC with t.p.m. $\mathcal{P}_{\omega_t}(\omega_{t+1})$).

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$$\text{subject to}\quad X_{s+1}=X_s+g_{s,\omega_{s+1}}(X_s,\alpha_s)\,,\quad s=t,\ldots,T\,, \tag{*}$$

$\alpha_t \overset{\text{def}}{=}$ the choice of control, $\omega_t \overset{\text{def}}{=}$ the realized uncertain state (MC with t.p.m. $\mathcal{P}_{\omega_t}(\omega_{t+1})$).

Let $V_{T+1,\omega_{t+1}}\equiv 0$ and let

$$V_{t,\omega_t}(x)=\max_{\alpha_t}\left(f_{t,\omega_t}(x,\alpha_t)+\mathsf{E}_t\left[V_{t+1,\omega_{t+1}}\left(x+g_{t,\omega_{t+1}}(x,\alpha_t)\right)\right]\right),\quad x\in\mathbb{R}\,,\quad t=0,1,\ldots T\,.$$

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$$\underset{\alpha_t,\ldots,\alpha_T}{\text{maximize}}\Big(f_{t,\omega_t}(X_t,\alpha_t)+\mathsf{E}_t\Big[\sum_{s=t+1}^{T} f_{s,\omega_s}(X_s,\alpha_s)\Big]\Big), \quad 0\le t\le T,$$

$$\text{subject to} \quad X_{s+1}=X_s+g_{s,\omega_{s+1}}(X_s,\alpha_s), \quad s=t,\ldots,T, \qquad (*)$$

$\alpha_t \overset{\text{def}}{=}$ the choice of control, $\omega_t \overset{\text{def}}{=}$ the realized uncertain state (MC with t.p.m. $\mathcal{P}_{\omega_t}(\omega_{t+1})$).

Let $V_{T+1,\omega_{t+1}} \equiv 0$ and let

$$V_{t,\omega_t}(x)=\max_{\alpha_t}\Big(f_{t,\omega_t}(x,\alpha_t)+\mathsf{E}_t\Big[V_{t+1,\omega_{t+1}}\big(x+g_{t,\omega_{t+1}}(x,\alpha_t)\big)\Big]\Big), \quad x\in\mathbb{R}, \quad t=0,1,\ldots T.$$

Let $\Phi_{s+1,\omega_{s+1}}\times\mathcal{P}_{\omega_s}(\omega_{s+1}) \overset{\text{def}}{=}$ the Lagrange multiplier attached to (*).

Decoupling Fields (an Illustration)

Without mean field interactions, a generic stochastic optimal control problem can be stated as:

$$\underset{\alpha_t,\dots,\alpha_T}{\text{maximize}}\Big(f_{t,\omega_t}(X_t,\alpha_t)+\mathsf{E}_t\Big[\sum\nolimits_{s=t+1}^{T} f_{s,\omega_s}(X_s,\alpha_s)\Big]\Big),\quad 0\le t\le T\,,$$

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Let $\Phi_{s+1,\omega_{s+1}}\times\mathcal{P}_{\omega_s}(\omega_{s+1})\overset{\text{def}}{\equiv}$ the Lagrange multiplier attached to (*).

Then $\Phi_{t+1,\omega_{t+1}}=\nabla V_{t+1,\omega_{t+1}}(X_{t+1})$ and one must solve (simultaneously for all t and all ω_t)

$$\mathcal{D}f_{t,\omega_t}(X_t,\alpha_t)+\mathsf{E}_t\big[\Phi_{t+1,\omega_{t+1}}\times\mathcal{D}g_{t,\omega_{t+1}}(X_t,\alpha_t)\big]=0\,,\qquad X_{t+1}=X_t+g_{t,\omega_{t+1}}(X_t,\alpha_t)\,,$$

$$\text{and}\quad \Phi_{t,\omega_t}\equiv\nabla V_{t,\omega_t}(X_t)=\nabla f_{t,\omega_t}(X_t,\alpha_t)+\mathsf{E}_t\Big[\Phi_{t+1,\omega_{t+1}}\times\big(1+\nabla g_{t,\omega_{t+1}}(X_t,\alpha_t)\big)\Big]\,.$$

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and $$\Phi_{t,\omega_t}\equiv\nabla V_{t,\omega_t}(X_t)=\nabla f_{t,\omega_t}(X_t,\alpha_t)+\mathsf{E}_t\Big[\Phi_{t+1,\omega_{t+1}}\times\big(1+\nabla g_{t,\omega_{t+1}}(X_t,\alpha_t)\big)\Big].$$

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Let $\boldsymbol{\Phi}_{s+1,\omega_{s+1}}\times\mathcal{P}_{\omega_s}(\omega_{s+1})\overset{\text{def}}{=}$ the Lagrange multiplier attached to (*).

Then $\boldsymbol{\Phi}_{t+1,\omega_{t+1}}=\nabla V_{t+1,\omega_{t+1}}(X_{t+1})$ and one must solve (simultaneously for all t and all ω_t)

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$$\text{and}\quad \boldsymbol{\Phi}_{t,\omega_t}\equiv\nabla V_{t,\omega_t}(X_t)=\nabla f_{t,\omega_t}(X_t,\alpha_t)+\mathsf{E}_t\left[\boldsymbol{\Phi}_{t+1,\omega_{t+1}}\times\left(1+\nabla g_{t,\omega_{t+1}}(X_t,\alpha_t)\right)\right].$$

N.B. Solving the optimization problem boils down to connecting $\boldsymbol{\Phi}_{t,\omega_t}$ with $\boldsymbol{\Phi}_{t+1,\omega_{t+1}}$, and, at the same time, connecting X_{t+1} with X_t.

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$\alpha_t \overset{\text{def}}{=}$ the choice of control, $\omega_t \overset{\text{def}}{=}$ the realized uncertain state (MC with t.p.m. $\mathcal{P}_{\omega_t}(\omega_{t+1})$).

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N.B. Solving the optimization problem boils down to connecting Φ_{t,ω_t} with $\Phi_{t+1,\omega_{t+1}}$, and, at the same time, connecting X_{t+1} with X_t. This process is backward-forward and, thus, non-programmable.

Decoupling Fields (an Illustration)

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At time $t = T$ set $\Phi_{T+1,\omega_{T+1}}(\cdot) \equiv \Phi_{T+2,\omega_{T+2}}(\cdot) \equiv 0$ and solve for $\alpha_T \equiv \alpha_{T,\omega_T}(x)$ from $\mathcal{D}f_{T,\omega_T}(x, \alpha_T) = 0$.

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Assuming that $\alpha_{t+1,\omega_{t+1}}(\cdot)$ and $\Phi_{t+2,\omega_{t+2}}(\cdot)$ are known functions, solve for $\alpha_t \equiv \alpha_{t,\omega_t}(x)$ from the equation

$$\mathcal{D} f_{t,\omega_t}(x,\alpha_t) + \mathsf{E}_t\Big[\Phi_{t+1,\omega_{t+1}}(x)\,\mathcal{D} g_{t,\omega_{t+1}}(x,\alpha_t)\Big] = 0\,,$$

in which

$$\Phi_{t+1,\omega_{t+1}}(x) \overset{\text{def}}{=} \nabla f_{t+1,\omega_{t+1}}(X^x_{t+1}, \alpha_{t+1,\omega_{t+1}}(X^x_{t+1})) + \mathsf{E}_{t+1}\Big[\Phi_{t+2,\omega_{t+2}}(X^x_{t+1})\big(1+\nabla g_{t+1,\omega_{t+2}}(X^x_{t+1},\alpha_{t+1,\omega_{t+1}}(X^x_{t+1}))\big)\Big]$$

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and $$X^x_{t+1} \stackrel{\text{def}}{=} x + g_{t,\omega_{t+1}}(x, \alpha_t)\,.$$

This gives the function $x \rightsquigarrow \alpha_{t,\omega_t}(x)$, and therefore also the function $x \rightsquigarrow \Phi_{t+1,\omega_{t+1}}(x)$ for $0 \le t < T$.

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After $\Phi_{t,\omega_t}(\cdot)$ and $\alpha_{t,\omega_t}(\cdot)$, $0 \le t \le T$, $\omega_t \in \Omega$, have been computed, assuming that $X_0 = x$, is given, the optimal path (X_t) can be constructed recursively, by stepping only *forward*, as

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N.B. Choosing $\{\omega_t :\ 0 \le t \le T\}$ to be i.i.d., $f_{t,\omega_t} \equiv f_t$ and $g_{t,\omega_t} \equiv g_t$ simplifies the picture:

$$\alpha_{t,\omega_t}(\cdot) = \alpha_t(\cdot)\,, \quad \Phi_{t,\omega_t}(\cdot) = \Phi_t(\cdot) \quad (\text{since } V_{t,\omega_t}(X_t) = V_t(X_t) \text{ in this case}).$$

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A generic stochastic optimal control problem in continuous time can be stated as

$$\underset{(\alpha_s,\, t\le s\le T)}{\text{maximize}}\, \mathsf{E}_t\left[\int_t^T f(X_s,\alpha_s,s)\,\mathrm{d}s + G(X_T,\alpha_T)\right]$$

$$\text{subject to}\quad X_s = X_t + \int_t^s \sigma(X_u,\alpha_u)\,\mathrm{d}W_u + \int_t^s b(X_u,\alpha_u)\,\mathrm{d}u\,,\quad t\le s\le T\,.$$

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As the shocks are i.i.d. the dual process and the control can be sought in the forms $\Phi(X_t,t)$ and $\alpha(X_t,t)$.

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At time $t = T$ solve for $\alpha_T \equiv \alpha(x,T)$ from $\mathcal{D}G(x,\alpha_T) = 0$ and set $\Phi(x,T) = \nabla G(x,\alpha_T(x,T))$.

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At time $t=T$ solve for $\alpha_T\equiv\alpha(x,T)$ from $\mathcal{D}G(x,\alpha_T)=0$ and set $\boldsymbol{\Phi}(x,T)=\nabla G(x,\alpha_T(x,T))$.
Then solve for $(x,t)\rightsquigarrow\boldsymbol{\Phi}(x,t)$ and $(x,t)\rightsquigarrow\alpha(x,t)$ from the "implicit" (and non-linear) backward PDE

$$-\partial\boldsymbol{\Phi}(x,t) = \nabla f(x,\alpha_t,t) + \nabla\boldsymbol{\Phi}(x,t)\times\Big(b(x,\alpha_t,t)+\sigma(x,\alpha_t,t)\,\nabla\sigma(x,\alpha_t,t)\Big) + \frac{1}{2}\nabla^2\boldsymbol{\Phi}(x,t)\,\sigma(x,\alpha_t,t)^2 + \boldsymbol{\Phi}(x,t)\,\nabla b(x,\alpha_t,t)$$

where $\alpha_t\equiv\alpha(x,t)\equiv\alpha\Big(x,\boldsymbol{\Phi}(x,t),\nabla\boldsymbol{\Phi}(x,t),t\Big)$ is determined implicitly from

$$\mathcal{D}f(x,\alpha_t,t)+\boldsymbol{\Phi}(x,t)\mathcal{D}b(x,\alpha_t,t)+\nabla\boldsymbol{\Phi}(x,t)\sigma(x,\alpha_t,t)\mathcal{D}\sigma(x,\alpha_t,t)=0\,.$$

Decoupling Fields (an Illustration)

The continuous-time analog of the strictly backward program is the well known decoupling field.

A generic stochastic optimal control problem in continuous time can be stated as

$$\underset{(\alpha_s,\, t\le s\le T)}{\text{maximize}}\ \mathsf{E}_t\left[\int_t^T f(X_s,\alpha_s,s)\,\mathrm{d}s + G(X_T,\alpha_T)\right]$$

$$\text{subject to}\quad X_s = X_t + \int_t^s \sigma(X_u,\alpha_u)\,\mathrm{d}W_u + \int_t^s b(X_u,\alpha_u)\,\mathrm{d}u\,,\quad t\le s\le T\,.$$

As the shocks are i.i.d. the dual process and the control can be sought in the forms $\Phi(X_t,t)$ and $\alpha(X_t,t)$. The goal is to construct the (deterministic) objects $\Phi(\cdot,\cdot)$ and $\alpha(\cdot,\cdot)$ (in "HJB style.").

At time $t=T$ solve for $\alpha_T\equiv\alpha(x,T)$ from $\mathcal{D}G(x,\alpha_T)=0$ and set $\Phi(x,T)=\nabla G(x,\alpha_T(x,T))$.

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$$\begin{aligned}-\partial\Phi(x,t) = \nabla f(x,\alpha_t,t) + \nabla\Phi(x,t)\times\big(b(x,\alpha_t,t)+\sigma(x,\alpha_t,t)\,\nabla\sigma(x,\alpha_t,t)\big)\\ +\frac{1}{2}\nabla^2\Phi(x,t)\,\sigma(x,\alpha_t,t)^2+\Phi(x,t)\,\nabla b(x,\alpha_t,t)\end{aligned}$$

where $\alpha_t\equiv\alpha(x,t)\equiv\alpha\big(x,\Phi(x,t),\nabla\Phi(x,t),t\big)$ is determined implicitly from

$$\mathcal{D}f(x,\alpha_t,t)+\Phi(x,t)\mathcal{D}b(x,\alpha_t,t)+\nabla\Phi(x,t)\sigma(x,\alpha_t,t)\mathcal{D}\sigma(x,\alpha_t,t)=0\,.$$

N.B. These relations provide a "connection between $\Phi(x,t)$ and $\Phi(x,t+\mathrm{d}t)$."

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The stipulation that the expected utility theory models reasonably well the way market participants respond to and settle (dis)agreements about uncertain outcomes may have to be revisited as well.