From quadratic Hawkes processes to rough volatility and Zumbach effect

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USC Colloqium in Mathematical Finance

February 7, 2020

Table of contents

Table of contents

First motivation : Volatility is ROUGH

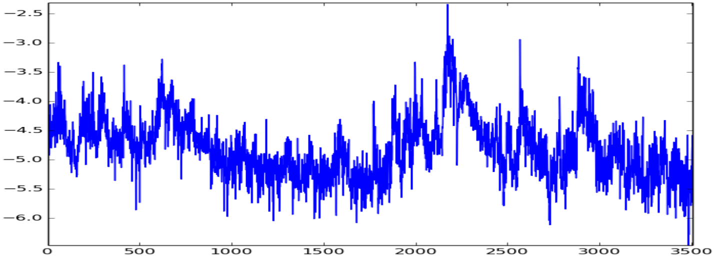


FIGURE – The log volatility of the S&P over about 10 years.

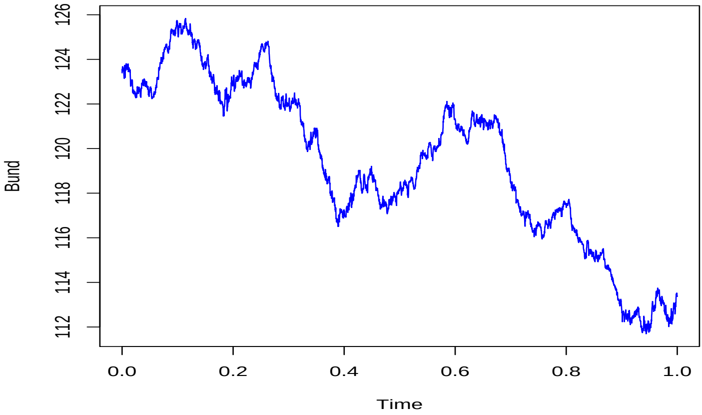
Our goal

Understanding volatility

- It is shown in Gatheral *et al.* that log-volatility time series behave in fact like a fractional Brownian motion, with Hurst parameter of order 0.1.
- More precisely, basically all the statistical stylized facts of volatility are retrieved when modeling it by a rough fractional Brownian motion.
- Such models also enable us to reproduce very well the behavior of the implied volatility surface, in particular the at-the-money skew (without jumps).
- Similar results on more than 6000 assets (Bennedsen *et al.*).
- **Why such behavior ?**
- We wish to explain the macroscopic dynamic of volatility from microstructural features of the asset.

Bund contract, 18 months, one data every hour

We want to understand :



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Bund, one hour, one data every second

Based on :

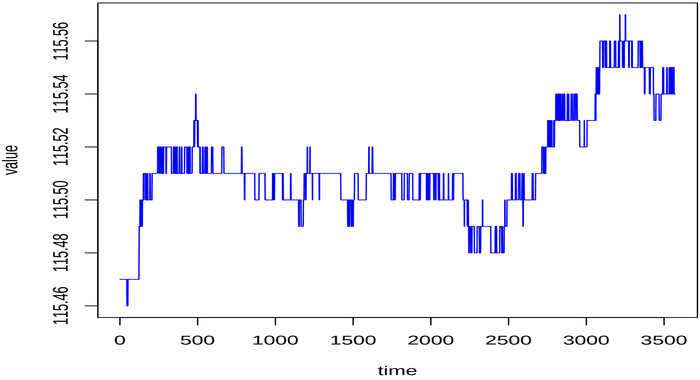

Table of contents

Building the model

Necessary conditions for a good microscopic price model

We want :

- A tick-by-tick model.
- A model reproducing the stylized facts of modern electronic markets in the context of high frequency trading.
- A model helping us to understand the rough dynamic of volatility from the high frequency behavior of market participants.

Building the model

Stylized facts 1-2

- Markets are highly endogenous, meaning that most of the orders have no real economic motivations but are rather sent by algorithms in reaction to other orders, see Bouchaud *et al.*, Filimonov and Sornette.
- Mechanisms preventing statistical arbitrages take place on high frequency markets, meaning that at the high frequency scale, building strategies that are on average profitable is hardly possible.

Building the model

Stylized facts 3-4

- There is some asymmetry in the liquidity on the bid and ask sides of the order book. In particular, a market maker is likely to raise the price by less following a buy order than to lower the price following the same size sell order, see Brennan *et al.*, Brunnermeier and Pedersen, Hendershott and Seasholes.
- A large proportion of transactions is due to large orders, called metaorders, which are not executed at once but split in time.

Building the model

Hawkes processes

- Our tick-by-tick price model is based on Hawkes processes in dimension two.
- A two-dimensional Hawkes process is a bivariate point process $(N_t^+, N_t^-)_{t\geq 0}$ taking values in $(\mathbb{R}^+)^2$ and with intensity $(\lambda_t^+, \lambda_t^-)$ of the form :

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\varphi_1(t-s) & \varphi_3(t-s)\\ \varphi_2(t-s) & \varphi_4(t-s)\end{pmatrix} . \begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix}.$$

Building the model

The microscopic price model

- Our model is simply given by

$$P_t = N_t^+ - N_t^-.$$

- N_t^+ corresponds to the number of upward jumps of the asset in the time interval $[0, t]$ and N_t^- to the number of downward jumps. Hence, the instantaneous probability to get an upward (downward) jump depends on the location in time of the past upward and downward jumps.
- By construction, the price process lives on a discrete grid.
- Statistical properties of this model have been studied in details.

Encoding the stylized facts about market microstructure

The right parametrization of the model

- Recall that

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\varphi_1(t-s) & \varphi_3(t-s)\\ \varphi_2(t-s) & \varphi_4(t-s)\end{pmatrix} \cdot \begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix}.$$

- High degree of endogeneity of the market $\rightarrow$ L^1 norm of the largest eigenvalue of the kernel matrix close to one.
- No arbitrage$\rightarrow$ $\varphi_1 + \varphi_3 = \varphi_2 + \varphi_4$.
- Liquidity asymmetry$\rightarrow$ $\varphi_3 = \beta\varphi_2$, with $\beta > 1$.
- Metaorders splitting$\rightarrow$ $\varphi_1(x),\ \varphi_2(x) \underset{x\to\infty}{\sim} K/x^{1+\alpha}$, $\alpha \approx 0.6$.

The scaling limit of the price model

Limit theorem

After suitable scaling in time and space, the long term limit of our price model satisfies the following rough Heston dynamics :

$$P_t = \int_0^t \sqrt{V_s} dW_s - \frac{1}{2} \int_0^t V_s ds,$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - V_s) ds + \frac{\lambda\nu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s,$$

with

$$d\langle W, B \rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}} dt.$$

The scaling limit of the price model

Comments on the theorem

- The Hurst parameter $H = \alpha - 1/2$.
- Hence stylized facts of modern market microstructure naturally give rise to fractional dynamics and leverage effect.
- One of the only cases of scaling limit of a non ad hoc "micro model" where leverage effect appears in the limit. Compare with Nelson's limit of GARCH models for example.
- Uniqueness of the limiting solution is a difficult result. The proof requires the use of recent results in SPDEs theory by Mytnik and Salisbury.
- Obtaining a non-zero starting value for the volatility is a tricky point. To do so, we in fact consider a time-dependent μ.

Table of contents

Light-Tailed case

Scaling Limits of nearly unstable Hawkes

- $\int_0^\infty s\phi(s)ds < \infty$.
- Intensity takes the form

$$\lambda_t = \mu + \int_0^t \phi(t-s)dN_s.$$

- Stable : fixed L^1 norm, get a deterministic asymptotic limit
- Nearly unstable : $||\phi||_1 \to 1$, corresponds to high market endogeneity
- Under suitable rescaling, limiting intensity :

$$X_t = \int_0^t (\mu - X_s)\frac{\lambda}{m}ds + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{X}_s dB_s.$$

Fat-Tailed case

Scaling Limits of nearly unstable Hawkes

- $f^{\alpha,\lambda} = \lambda x^{\alpha-1} E_{\alpha,\alpha}(-\lambda x^{\alpha})$.
- $F^{\alpha,\lambda}(t) = \int_0^t f^{\alpha,\lambda}(s)ds$.
- $\phi(x)$ of the form $x^{-(1+\alpha)}$, $\alpha \in (0,1)$.
- Limiting intensity : fractional CIR :

$$X_t = F_t^{\alpha,\lambda} + \int_0^t f^{\alpha,\lambda}(t-s)\sqrt{X}_s dB_s.$$

- Link between persistence properties and irregularity of limiting process

Table of contents

Further aspects of volatility

Zumbach effect (Zumbach *et al.*) : description

- Feedback of price returns on volatility.
- Price trends induce an increase of volatility.
- In the literature (notably works by J.P. Bouchaud and co-authors), a way to reinterpret the Zumbach effect is to consider that the predictive power of past squared returns on future volatility is stronger than that of past volatility on future squared returns.
- To check this on data, one typically shows that the covariance between past squared price returns and future realized volatility (over a given duration) is larger than that between past realized volatility and future squared price returns.
- We refer to this version of Zumbach effect as *weak Zumbach effect*.

Further aspects of volatility

Weak and strong Zumbach effect

- It is shown in Gatheral *et al.* that the rough Heston model reproduces the weak form of Zumbach effect.
- However, it is not obtained through feedback effect, which is the motivating phenomenon in the original paper by Zumbach. It is only due to the dependence between price and volatility induced by the correlation of the Brownian motions driving their dynamics.
- In particular in the rough Heston model, the conditional law of the volatility depends on the past dynamic of the price only through the past volatility.
- We speak about *strong Zumbach effect* when the conditional law of future volatility depends not only on past volatility trajectory but also on past returns.

A convenient microscopic model encoding Zumbach effect

Quadratic Hawkes processes

- Inspired by Blanc *et al.*, we model high frequency prices using quadratic Hawkes processes.
- Jump sizes of the price P_t are i.i.d taking values -1 and 1 with probability $1/2$ and jump times are those of a point process N_t with intensity

$$\lambda_t = \mu + \int_0^t \phi(t-s)\mathrm{d}N_s + Z_t^2, \text{ with } Z_t = \int_0^t k(t-s)\mathrm{d}P_s.$$

- The component Z_t is a moving average of past returns.
- If the price has been essentially trending in the past, Z_t is large leading to high intensity. On the contrary if it has been oscillating, Z_t is close to zero and there is no feedback from the returns on the volatility. Hence Z_t is a (strong) Zumbach term.
- Stability condition for such model : $\|\phi\|_1 + \|k\|_2^2 < 1$.

Scaling limits

Purely quadratic case

- When $\phi = 0$ is equal to zero, choosing appropriate scaling parameters, we obtain the following limiting model : $\mathrm{d}\hat{P}_t = \sqrt{V_t}\mathrm{d}B_t$ with

$$V_t = \mu + Z_t^2, \quad Z_t = \sqrt{\gamma}\int_0^t k(t-s)\mathrm{d}\hat{P}_s.$$

- In contrast to the purely linear case, we do not need any sort of near instability so that a stochastic volatility model arises at the scaling limit.
- Not surprising since quadratic Hawkes models share many similarities with GARCH and QARCH models
- GARCH-like processes lead to stochastic volatility at the scaling limit without any degeneracy in their parameters
- The strong Zumbach effect is naturally encoded since the volatility is a functional of past price returns through Z.

Scaling limits

Purely quadratic case (2)

- The quadratic feedback of price returns on volatility implies that V_t is of super-Heston type (essentially log-normal here).
- This can be seen for example when $\mu = 0$:

$$Z_t = \sqrt{\gamma} \int_0^t k(t-s)|Z_s| \mathrm{d}B_s.$$

- Taking for example $k = f^{H+1/2,\lambda}$ for $H \in (0, 1/2)$ and $\lambda > 0$ with $f^{\alpha,\lambda}$ the Mittag-Leffler function, volatility has Hölder regularity $H - \varepsilon$.
- Roughness is generated by the behavior in 0 of k.
- From a natural microscopic dynamic, we obtain a super-Heston rough volatility model with strong Zumbach effect at the macroscopic limit.
- Quite similar result is obtained when $\phi \neq 0$ (additional drift term in the dynamic) provided we do not enter the near instability regime.

Scaling limits

Nearly unstable regime

- We know that we can generate roughness from the linear part only in the near instability regime.
- In this regime, assuming $\phi(x)$ behaves as $x^{-(1+\alpha)}$ as x goes to infinity, we prove that at the limit : $\mathrm{d}\hat{P}_t = \sqrt{V_t}\mathrm{d}B_t^{(1)}$ with

$$
\begin{aligned}
V_t &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\big(\theta^0(s) + Z_s^2 - V_s\big)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\eta\sqrt{V_s}\mathrm{d}B_s^{(2)} \\
Z_t &= \int_0^t k(t-s)\sqrt{V_s}\mathrm{d}B_s^{(1)}.
\end{aligned}
$$

Scaling limits

Nearly unstable regime (2)

- As in the linear case, the near instability condition leads to appearance of a second Brownian motion driving a rough Heston type term.
- We see that the strong Zumbach effect is still reproduced thanks to the Z_t^2 term which is here convoluted with a power-law kernel.
- Roughness comes from the tail of ϕ.

Scaling limits

Nearly unstable regime (3)

- When k is regular, we have (up to a finite variation term)

$$V_t = \int_0^t f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}}\sqrt{V_s}\mathrm{d}B_s^{(1)} + \int_0^t F^{\alpha,\lambda}(t-s)k(0)Z_s\sqrt{V_s}\mathrm{d}B_s^{(2)}.$$

- Thus as in the stable case the quadratic feedback term in the volatility dynamic induces a super-Heston type rough volatility model.

Conclusions

From Hawkes processes to rough volatility

- Rough volatility is a universal phenomenon.
- Hawkes processes are very suitable models for the dynamics of high frequency prices.
- Under very natural specifications, they give rise to rough volatility at the macroscopic scale.
- In fact one can even show that rough volatility is implied by a no-statistical arbitrage type condition only.