Weakly interacting particle systems on graphs: from dense to sparse

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Overview

1. Networks of interacting stochastic processes
2. Local convergence
3. Conditional independence
4. Local dynamics on infinite regular trees

Networks of interacting stochastic processes

Given a finite connected graph $G = (V, E)$, write

d_v = degree of vertex v, and $u \sim v$ if $(u, v) \in E$.

Each node $v \in V$ has a particle whose stochastic evolution depends only on its own state and that of nearest neighbors

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For example, as a discrete-time Markov chain:

$$X_v(t+1) = F\Big(X_v(t), (X_u(t))_{u \sim v}, \xi_v(t+1)\Big).$$

- State space S
- Continuous transition function F
- Independent noises $\xi_v(t)$, $v \in V$, $t = 0, 1, \ldots$

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Arise in: probabilistic cellular automaton, synchronous Markov chain, simultaneous updating
Examples: Voter model, contact process, exclusion processes

Example: Contact process

State space $S = \{0, 1\} = \{\text{off/healthy}, \text{on/infected}\}$. Parameters $p, q \in [0, 1]$.

Transition rule: At time t, if particle v is at...

- state $X_v(t) = 1$, it switches to $X_v(t+1) = 0$ w.p. q,
- state $X_v(t) = 0$, it switches to $X_v(t+1) = 1$ w.p.

$$\frac{p}{d_v} \sum_{u \sim v} X_u(t).$$

Networks of interacting diffusions

Given a finite connected graph $G = (V, E)$, write
d_v = degree of vertex v, and $u \sim v$ if $(u, v) \in E$.

Or as a diffusion:

$$dX_v(t) = \frac{1}{d_v} \sum_{u \sim v} b(X_v(t), X_u(t))dt + dW_v(t),$$

where $(W_v)_{v \in V}$ are independent Brownian motions.

For concreteness, assume b is Lipschitz throughout, and initial states $(X_v(0))_{v \in V}$ are i.i.d. and square-integrable.

Networks of interacting stochastic processes

Key questions

Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, how can we describe the limiting behavior of...

- the dynamics of a "typical" or fixed particle $X_v(t)$, $t \in [0, T]$?
- the empirical distribution of particles $\frac{1}{|V_n|} \sum_{v \in V_n} \delta_{X_v(t)}$?

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Mean field as a special case

If G_n is the complete graph on n vertices, we are in the mean field (McKean-Vlasov) setting.

The mean field case (McKean-Vlasov 1966)

Particles $i = 1, \dots, n$ interact according to

$$dX_t^i = \frac{1}{n} \sum_{k=1}^{n} b(X_t^i, X_t^k) dt + dW_t^i,$$

where $W^1, \dots, W^n$ are independent Brownian, $(X_0^1, \dots, X_0^n)$ i.i.d.

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This can be reformulated as

$$dX_t^i = B(X_t^i, \bar{\mu}_t^n) dt + dW_t^i, \qquad \bar{\mu}_t^n = \frac{1}{n} \sum_{k=1}^{n} \delta_{X_t^k},$$

where

$$B(x, m) = \int_{\mathbb{R}^d} b(x, y)\, m(dy).$$

Also referred to as weakly interacting diffusions.

The mean field case, law of large numbers

Theorem (Sznitman '91, etc.)

$(\bar{\mu}^n_t)_{t\in[0,T]}$ *converges in probability to the unique solution* $(\mu_t)_{t\in[0,T]}$ *of the McKean-Vlasov equation*

$$dX_t = B(X_t, \mu_t)\, dt + dW_t, \qquad \mu_t = \mathrm{Law}(X_t).$$

Moreover, the particles become asymptotically independent (propagation of chaos). Precisely, for fixed k,

$$(X^1, \dots, X^k) \Rightarrow \mu^{\otimes k}, \qquad \text{as } n \to \infty.$$

Mean field limits for sequences of dense graphs

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Theorem (Bhamidi-Budhiraja-W. '16)

Suppose $G_n = G(n, p_n)$ is Erdős-Rényi, with $np_n \to \infty$. Then everything behaves like in the mean field case.

See also: Delattre-Giacomin-Luçon '16, Coppini-Dietert-Giacomin '18.

Theorem (Budhiraja-Mukherjee-W. '17)

Suppose $G_n = G(n, p_n)$ is Erdős-Rényi, with $np_n \to \infty$. For the supermarket model (i.e. the power-of-d load balancing scheme), everything behaves like in the mean field case.

Beyond mean field limits

Key questions

Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, how can we describe the limiting behavior of...

- a "typical" or fixed particle $X_v(t)$?
- the empirical distribution of particles $\frac{1}{|V_n|} \sum_{v \in V_n} \delta_{X_v(t)}$?

Observation: $G_n = G(n, p_n)$ is Erdős-Rényi, $np_n \approx$ average degree $\to \infty$, the graphs are dense.

Our focus: The sparse regime, where degrees do not diverge.
How does the $n \to \infty$ limit reflect the graph structure?

Example: Erdős-Rényi $G(n, p_n)$ with $np_n \to p \in (0, \infty)$.

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How does the $n \to \infty$ limit reflect the graph structure?

Example: Erdős-Rényi $G(n, p_n)$ with $np_n \to p \in (0, \infty)$.

Questions: (Q1) Does the whole system admit a scaling limit?
(Q2) Is there a nice autonomous description of the limiting dynamics?

Example: Detering-Fouque-Ichiba '18 treats directed cycle graph.

Beyond mean field limits: Key ingredients

Sequence of sparse graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$,

$$X_v(t+1) = F(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)), \quad t = 0, 1, \dots,$$
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(Q1) Does the whole system admit a scaling limit?
(A1) Yes, using generalized notion of local (weak) convergence

(Q2) Is there a nice autonomous description of the limiting dynamics?
(A2) For regular trees or Galton-Watson trees, Yes: due to a certain space-time Markov random-field property

Local convergence of graphs

Idea: Encode sparsity via local convergence of graphs.
(a.k.a. Benjamini-Schramm convergence, see Aldous-Steele '04)

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Definition: A graph $G = (V, E, \o)$ is assumed to be rooted, finite or countable, locally finite, and connected.

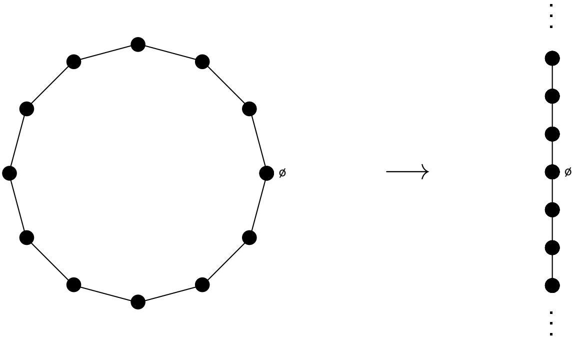
Definition: Rooted graphs G_n converge locally to G if:

$$\forall k, \exists N \text{ s.t. } B_k(G) \cong B_k(G_n) \text{ for all } n \geq N,$$

where $B_k(\cdot)$ is ball of radius k at root (with respect to the graph distance), and $\cong$ means isomorphism.

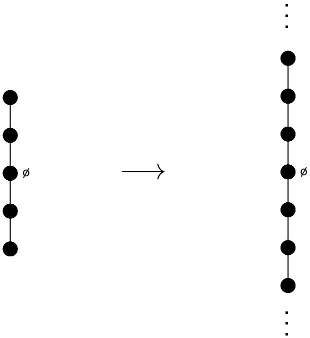
Examples of local convergence

1. Cycle graph converges to infinite line



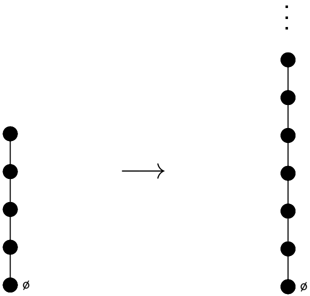
Examples of local convergence

2. Line graph converges to infinite line



Examples of local convergence

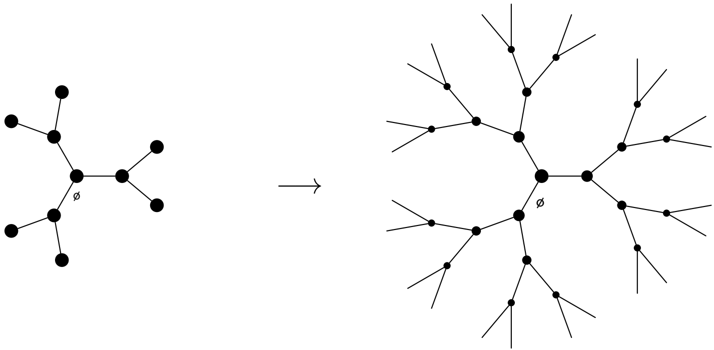
3. Line graph rooted at end converges to semi-infinite line



Examples of local convergence

4. Finite to infinite d-regular trees

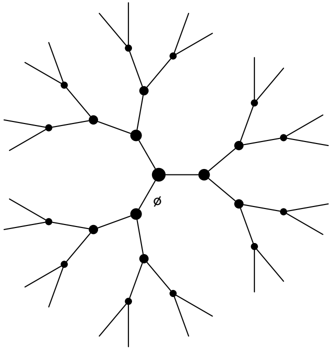
(A graph is d-regular if every vertex has degree d.)



Examples of local convergence

5. Uniformly random regular graph to infinite regular tree

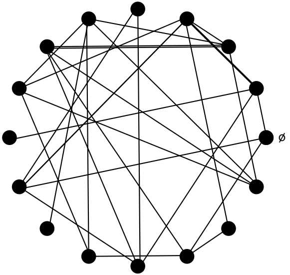
Fix d. Among all d-regular graphs on n vertices, select one uniformly at random. Place the root at a (uniformly) random vertex. When $n \to \infty$, this converges (in law) to the infinite d-regular tree. (Bollobás '80) $\longrightarrow$



Examples of local convergence

6. Erdős-Rényi to Galton-Watson(Poisson)

If $G_n = G(n, p_n)$ with $np_n \to p \in (0, \infty)$, then G_n converges in law to the Galton-Watson tree with offspring distribution $\mathrm{Poisson}(p)$.



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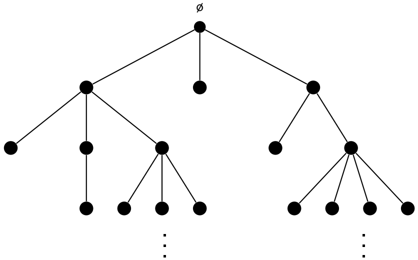

7. Configuration model to unimodular Galton-Watson

If G_n is drawn from the configuration model on n vertices with degree distribution $\rho \in \mathcal{P}(\mathbb{N})$, then G_n converges in law to the unimodular Galton-Watson tree $\mathrm{UGW}(\rho)$.

- Construct $\mathrm{UGW}(\rho)$ by letting root have ρ-many children, and each child thereafter has $\widehat{\rho}$-many children, where

$$\widehat{\rho}(n) = \frac{(n+1)\rho(n+1)}{\sum_k k\rho(k)}.$$

- **Example 1**: $\rho = \mathrm{Poisson}(p) \implies \widehat{\rho} = \mathrm{Poisson}(p)$.
- **Example 2**: $\rho = \delta_d \implies \widehat{\rho} = \delta_{d-1}$, so $\mathrm{UGW}(\delta_d)$ is the (deterministic) infinite d-regular tree.

Local convergence of marked graphs

Recall: $G_n = (V_n, E_n, \o_n)$ converges locally to $G = (V, E, \o)$ if

$$\forall k, \exists N \text{ s.t. } B_k(G) \cong B_k(G_n) \text{ for all } n \geq N.$$

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Definition: With G_n, G as above: Given a metric space (S, d_S) and a sequence $\boldsymbol{x}^n = (x_v^n)_{v \in G_n} \in S^{G_n}$, say that $(G_n, \boldsymbol{x}^n)$ converges locally to $(G, \boldsymbol{x})$ if

$$\forall k, \epsilon > 0 \; \exists N \text{ s.t. } \forall n \geq N \; \exists \varphi : B_k(G_n) \to B_k(G) \text{ isomorphism}$$
$$\text{s.t. } \max_{v \in B_k(G_n)} d_S(x_v^n, x_{\varphi(v)}) < \epsilon.$$

Lemma

The set $\mathcal{G}_*[S]$ of (isomorphism classes of) $(G, \boldsymbol{x})$ admits a Polish topology compatible with the above convergence.

Local convergence of marked graphs

Recall: Particle system on a rooted locally finite graph $G = (V, E, \varnothing)$:

$$X_v(t+1) = F(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)), \quad t = 0, 1, \ldots,$$

$$dX_v^G(t) = \frac{1}{d_v}\sum_{u\sim v} b(X_v^G(t), X_u^G(t))dt + dW_v(t), \quad t \geq 0.$$

Theorem

If $G_n \to G$ locally, then (G_n, X^{G_n}) converges in law to (G, X^G) in $\mathcal{G}_*[C(\mathbb{R}_+;\mathbb{R}^d)]$ (or $\mathcal{G}_*[(\mathbb{R}^d)^\infty]$ for Markov chains). Valid for random graphs too.

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In particular, the root particles converge: $X_{\o_n}^{G_n} \Rightarrow X_{\o}^G$ in $C(\mathbb{R}_+; \mathbb{R}^d)$.

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Empirical measure convergence is harder. If $G_n \sim G(n, p_n)$, $np_n \to p \in (0, \infty)$, then

$$\frac{1}{|G_n|}\sum_{v\in G_n} \delta_{X_v^{G_n}} \Rightarrow \mathrm{Law}(X_\varnothing^T), \text{ in } \mathcal{P}(C(\mathbb{R}_+;\mathbb{R}^d)),$$

where $T \sim \mathrm{GW}(\mathrm{Poisson}(p))$.

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(Q1): Does the whole system admit a scaling limit?

Theorem (Answer)

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Recall (Q2) Is there a nice autonomous description of the limiting dynamics for regular trees or Galton-Watson trees?
Goal: For unimodular Galton-Watson trees, find autonomous dynamics for the root neighborhood particles.

Markov random field

Notation: For a set A of vertices in a graph $G = (V, E)$, define

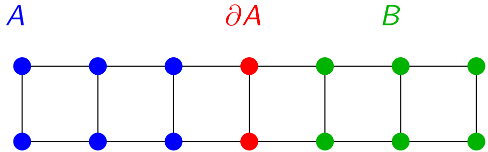
$$\text{Boundary: } \partial A = \{u \in V \backslash A : \exists u \in A \text{ s.t. } u \sim v\}.$$

Definition: A family of random variables $(Y_v)_{v \in V}$ is a Markov random field (MRF) if

$$(Y_v)_{v \in A} \perp (Y_v)_{v \in B} \;\mid\; (Y_v)_{v \in \partial A},$$

for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial A) = \emptyset$.

Example:



Searching conditional independence property

$$X_v(t+1) = F(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)), \quad t = 0, 1, \ldots,$$
$$dX_v(t) = \frac{1}{d_v}\sum_{u\sim v} b(X_v(t), X_u(t))dt + dW_v(t), \quad t \geq 0.$$

Assume the initial states $(X_v(0))_{v\in V}$ are i.i.d.

Question #1: (Spatial MRF)

For each time t, do the particle positions $(X_v(t))_{v\in V}$ form a Markov random field?

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Question #2: (Space-time MRF)

For each time t, do the particle trajectories $(X_v[t])_{v\in V}$ form a Markov random field? Here $x[t] = (x(s), s \in [0, t])$. Namely for any finite $A \subset V$ and $t > 0$, is $\boldsymbol{X}_A[t] \perp \boldsymbol{X}_{V\setminus(A\cup\partial A)}[t] \mid \boldsymbol{X}_{\partial A}[t]$?

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Answer #2:

NO

Searching conditional independence property

Counter-example: Consider three real-valued interacting Markov chains $\boldsymbol{X}(k) = (X_i(k) : i = 1, 2, 3)$ on a line:

$$\boldsymbol{X}(k+1) = B\boldsymbol{X}(k) + \boldsymbol{\xi}(k), \quad \boldsymbol{X}(0) = \boldsymbol{\xi}(0),$$

where $\xi_i(k)$ is i.i.d. $\mathcal{N}(0,1)$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Then one can get

$$(\boldsymbol{X}(1), X_2(0)) \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 3 & 2 & 1 & 1 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right),$$

$$Var(X_1(1), X_3(1) \mid X_2(1), X_2(0)) = \begin{bmatrix} 5/3 & -1/3 \\ -1/3 & 5/3 \end{bmatrix}.$$

So $X_1(1)$ is not independent of $X_3(1)$ given $(X_2(1), X_2(0))$.

Second-order Markov random field

Notation: For a set A of vertices in a graph $G = (V, E)$, define

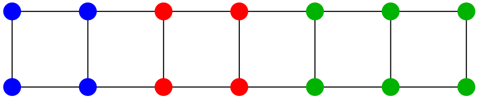
$$\text{Double-boundary: } \partial^2 A = \partial A \cup \partial(A \cup \partial A).$$

Definition: A family of random variables $(Y_v)_{v \in V}$ is a 2nd-order Markov random field if

$$(Y_v)_{v \in A} \perp (Y_v)_{v \in B} \ \mid \ (Y_v)_{v \in \partial^2 A},$$

for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Example: A $\partial^2 A$ B



Searching conditional independence property

$$X_v(t+1) = F(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)), \quad t = 0, 1, \dots,$$
$$dX_v(t) = \frac{1}{d_v}\sum_{u\sim v} b(X_v(t), X_u(t))dt + dW_v(t), \quad t \geq 0.$$

Assume the initial states $(X_v(0))_{v\in V}$ are i.i.d.

Question #3: (Spatial second-order MRF)

For each time t, do the particle positions $(X_v(t))_{v\in V}$ form a second-order Markov random field? Namely for any finite $A \subset V$ and $t > 0$, is

$$\boldsymbol{X}_A(t) \perp \boldsymbol{X}_{V\setminus(A\cup\partial^2 A)}(t) \mid \boldsymbol{X}_{\partial^2 A}(t)?$$

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Answer #3:

NO

Searching conditional independence property

Counter-example: Consider four real-valued interacting Markov chains $\boldsymbol{X}(k) = (X_i(k) : i = 1, 2, 3, 4)$ on a line:

$$\boldsymbol{X}(k+1) = B\boldsymbol{X}(k) + \boldsymbol{\xi}(k), \quad \boldsymbol{X}(0) = \boldsymbol{\xi}(0),$$

where $\xi_i(k)$ is i.i.d. $\mathcal{N}(0,1)$ and $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Then one can get

$$\boldsymbol{X}(2) \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 12 & 14 & 10 & 4 \\ 14 & 22 & 18 & 10 \\ 10 & 18 & 22 & 14 \\ 4 & 10 & 14 & 12 \end{bmatrix}\right),$$

$$Var(X_1(2), X_4(2) \mid X_2(2), X_3(2)) = \begin{bmatrix} 14/5 & -6/5 \\ -6/5 & 14/5 \end{bmatrix}.$$

So $X_1(2)$ is not independent of $X_4(2)$ given $(X_2(2), X_3(2))$.

Searching conditional independence property

$$X_v(t+1) = F(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)), \quad t = 0, 1, \dots,$$
$$dX_v(t) = \frac{1}{d_v}\sum_{u\sim v} b(X_v(t), X_u(t))dt + dW_v(t), \quad t \geq 0.$$

Assume the initial states $(X_v(0))_{v\in V}$ are i.i.d.

Question #4: (Space-time second order MRF)

For each time t, do the particle trajectories $(X_v[t])_{v\in V}$ form a second-order Markov random field? Here $x[t] = (x(s), s \in [0, t])$.

Searching conditional independence property

$$X_v(t+1) = F(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)), \quad t = 0, 1, \dots,$$

$$dX_v(t) = \frac{1}{d_v} \sum_{u \sim v} b(X_v(t), X_u(t))dt + dW_v(t), \quad t \geq 0.$$

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Question #4: (Space-time second order MRF)

For each time t, do the particle trajectories $(X_v[t])_{v\in V}$ form a second-order Markov random field? Here $x[t] = (x(s), s \in [0, t])$.

Answer #4: YES

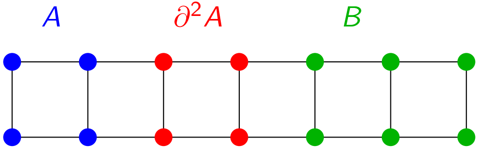
Theorem: (Lacker, Ramanan, W. '18)

$$\boldsymbol{X}_A[t] \perp \boldsymbol{X}_{V\setminus(A\cup\partial^2 A)}[t] \mid \boldsymbol{X}_{\partial^2 A}[t]$$

for all finite $A \subset V$ and $t > 0$.
In fact, suffices for $(X_v(0))_{v\in V}$ to form a second-order MRF.

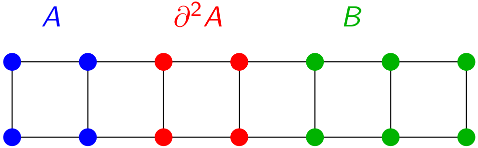
Searching conditional independence property

Intuition:



Searching conditional independence property

Intuition:

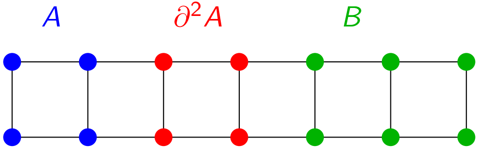


Proof idea:

Markov chain: Use induction and properties of conditional independence.

Searching conditional independence property

Intuition:

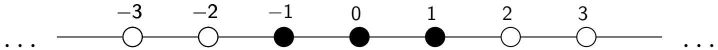


Proof idea:

Markov chain: Use induction and properties of conditional independence.

Diffusion: Use Girsanov to identify the density of $(X_v[t])_{v \in V}$ w.r.t. Wiener measure, and study how it factorizes. Use Hammersley-Clifford theorem.

Infinite line graph: Markov chain



Particle system on infinite line graph, $i \in \mathbb{Z}$:

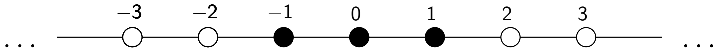
$$X_i(t+1) = F\Big(X_i(t), X_{i-1}(t), X_{i+1}(t), \xi_i(t+1)\Big)$$

Assume:

- F is symmetric in neighbors: $F(x, y, z, \xi) = F(x, z, y, \xi)$.

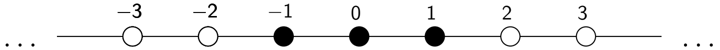
Goal: Find an autonomous stochastic process (Y_{-1}, Y_0, Y_1) which agrees in law with (X_{-1}, X_0, X_1).

Local dynamics for line graph: Markov chain



- Start with $(Y_{-1}, Y_0, Y_1)(0) = (X_{-1}, X_0, X_1)(0)$.

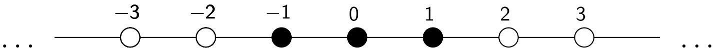
Local dynamics for line graph: Markov chain



- Start with $(Y_{-1}, Y_0, Y_1)(0) = (X_{-1}, X_0, X_1)(0)$.
- Given $(Y_{-1}, Y_0, Y_1)[t]$ at time t, let

$$\gamma_t(\cdot \mid y_0[t], y_1[t]) = \mathrm{Law}\Big(Y_{-1}(t) \mid Y_0[t] = y_0[t], Y_1[t] = y_1[t] \Big).$$

Local dynamics for line graph: Markov chain



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- Given $(Y_{-1}, Y_0, Y_1)[t]$ at time t, let

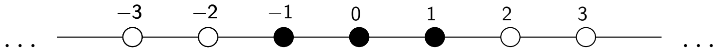
$$\gamma_t(\cdot \mid y_0[t], y_1[t]) = \mathrm{Law}\Big(Y_{-1}(t) \mid Y_0[t] = y_0[t],\, Y_1[t] = y_1[t] \Big).$$

- Independently sample ghost particles $Y_{-2}(t)$ and $Y_2(t)$ so that

$$\mathrm{Law}\Big(Y_{-2}(t) \mid Y_{-1}[t],\, Y_0[t],\, Y_1[t] \Big) = \gamma_t(\cdot \mid Y_{-1}[t],\, Y_0[t])$$

Here the conditional independence is used.

Local dynamics for line graph: Markov chain



- Start with $(Y_{-1}, Y_0, Y_1)(0) = (X_{-1}, X_0, X_1)(0)$.
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- Independently sample ghost particles $Y_{-2}(t)$ and $Y_2(t)$ so that

$$\mathrm{Law}\Big(Y_{-2}(t) \mid Y_{-1}[t], Y_0[t], Y_1[t] \Big) = \gamma_t(\cdot \mid Y_{-1}[t], Y_0[t])$$

  Here the conditional independence is used.
- Sample new noises $(\xi_{-1}, \xi_0, \xi_1)(t+1)$ independently, and update:

$$Y_i(t+1) = F\Big(Y_i(t), Y_{i-1}(t), Y_{i+1}(t), \xi_i(t+1) \Big), \quad i = -1, 0, 1$$

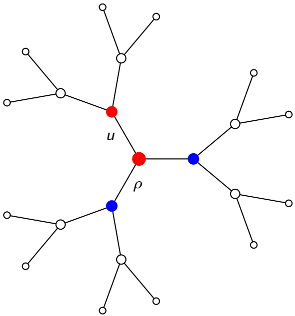
Infinite d-regular trees: Markov chain

Autonomous dynamics for root particle and its neighbors,

$$X_\rho(t),\ (X_v(t))_{v\sim\rho},$$

involving conditional law of $d-1$ children given root and one other child u:

$$\mathrm{Law}((X_v)_{v\sim\rho,\, v\neq u}\,|\,X_\rho,\, X_u)$$



$d=3$

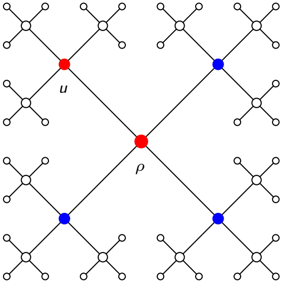
Infinite d-regular trees: Markov chain

Autonomous dynamics for root particle and its neighbors,

$$X_\rho(t),\ (X_v(t))_{v\sim\rho},$$

involving conditional law of $d-1$ children given root and one other child u:

$$\mathrm{Law}\big((X_v)_{v\sim\rho,\, v\neq u} \,|\, X_\rho,\ X_u\big)$$



$d = 4$

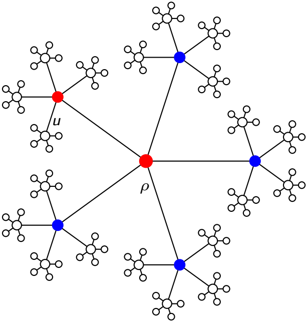
Infinite d-regular trees: Markov chain

Autonomous dynamics for root particle and its neighbors,

$$X_\rho(t),\ (X_v(t))_{v\sim\rho},$$

involving conditional law of $d-1$ children given root and one other child u:

$$\mathrm{Law}\big((X_v)_{v\sim\rho,\,v\neq u}\,|\,X_\rho,\ X_u\big)$$

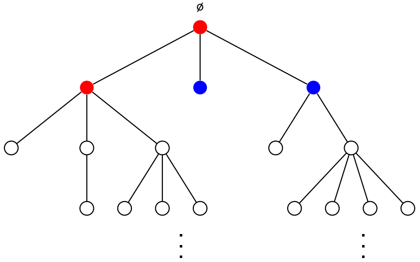


$d = 5$

Unimodular Galton-Watson trees: Markov chain

Autonomous dynamics for root & first generation involving conditional law of 1st-generation given root and one child.

Condition on tree structure as well!



Example: Linear Gaussian dynamics

State space $\mathbb{R}$, noises $\xi_v(t)$ are independent standard Gaussian.

$$X_v(t+1) = aX_v(t) + b\sum_{u\sim v} X_u(t) + c + \xi_v(t+1)$$
$$X_v(0) = \xi_v(0), \qquad a, b, c \in \mathbb{R}$$

$\rightsquigarrow$ conditional laws are all Gaussian

Proposition

Suppose the graph G is an infinite d-regular tree, $d > 2$. Simulating local dynamics for one particle up to time t is $O(t^2 d^2)$.

Compare: Simulation using infinite tree is $O((d-1)^{t+1})$.

Example: Contact process

Each particle is either 1 or 0. Parameters $p, q \in [0, 1]$.

Transition rule: At time t, if particle v...

- is at state $X_v(t) = 1$, it switches to $X_v(t+1) = 0$ w.p. q,
- is at state $X_v(t) = 0$, it switches to $X_v(t+1) = 1$ w.p.

$$\frac{p}{d_v} \sum_{u \sim v} X_u(t),$$

where d_v = degree of vertex v.

How well do local approximation and mean field approximation do?

Example: Contact process

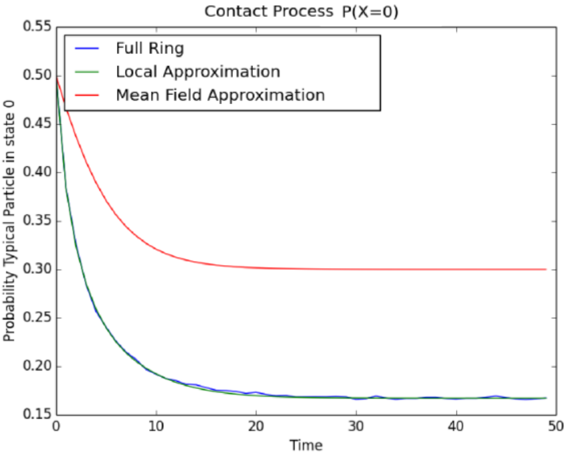


Figure: Infinite 2-regular tree (line), $p = 2/3$, $q = 0.1$

Credit: Ankan Ganguly & Mitchell Wortsman, Brown University

Local dynamics for diffusions

An additional ingredient: A projection/mimicking lemma (Brunick-Shreve '13; Gyongy '86)

On some $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, let W be $\mathbb{F}$-Brownian. Let $(b(t))$ be $\mathbb{F}$-adapted and square-integrable, with

$$dX(t) = b(t)dt + dW(t).$$

Define (by optional projection)

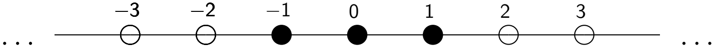
$$B(t,x) = \mathbb{E}[b(t) \,|\, X[t] = x[t]].$$

Then there is a weak solution Y of

$$dY(t) = B(t, Y)dt + d\widetilde{W}(t)$$

such that $Y \stackrel{d}{=} X$.

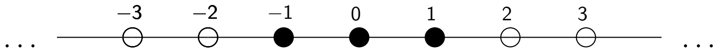
Local dynamics for line graph: diffusions



Particle system on infinite line graph, $i \in \mathbb{Z}$:

$$dX_t^i = \frac{1}{2}\left(b(X_t^i, X_t^{i-1}) + b(X_t^i, X_t^{i+1})\right) dt + dW_t^i$$

Local dynamics for line graph: diffusions



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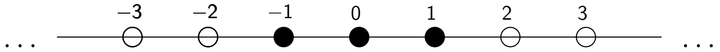
$$dX_t^i = \frac{1}{2}\left(b(X_t^i, X_t^{i-1}) + b(X_t^i, X_t^{i+1})\right) dt + dW_t^i$$

Local dynamics:

$$\begin{aligned}
dY_t^1 &= \frac{1}{2}\left(b(Y_t^1, Y_t^0) + \langle \gamma_t(Y^1, Y^0), b(Y_t^1, \cdot)\rangle\right) dt && + dW_t^1 \\
dY_t^0 &= \frac{1}{2}\left(b(Y_t^0, Y_t^{-1}) + b(Y_t^0, Y_t^1)\right) dt && + dW_t^0 \\
dY_t^{-1} &= \frac{1}{2}\left(\langle \gamma_t(Y^{-1}, Y^0), b(Y_t^{-1}, \cdot)\rangle + b(Y_t^{-1}, Y_t^0)\right) dt && + dW_t^{-1} \\
\gamma_t(y^0, y^{-1}) &= \mathrm{Law}\left(Y_t^1 \mid Y_{\cdot\wedge t}^0 = y_{\cdot\wedge t}^0,\ \ Y_{\cdot\wedge t}^{-1} = y_{\cdot\wedge t}^{-1}\right)
\end{aligned}$$

Thm: Uniqueness in law & $(Y^{-1}, Y^0, Y^1) \stackrel{d}{=} (X^{-1}, X^0, X^1)$.

Local dynamics for line graph: diffusions



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\end{aligned}$$

Thm: Uniqueness in law & $(Y^{-1}, Y^0, Y^1) \overset{d}{=} (X^{-1}, X^0, X^1)$.

Analogous local dynamics hold for infinite d-regular trees and unimodular Galton-Watson trees

Summary

Theorem 1

If finite graph sequence converges locally to infinite graph, then particle systems converge locally as well.

Theorem 2

Root neighborhood particles in a unimodular Galton-Watson tree admit well-posed local dynamics.

Corollary: If finite graph sequence converges locally to a unimodular Galton-Watson tree, then root neighborhood particles converge to unique solution of local dynamics.

Thank you!

References

[1] S. Bhamidi, A. Budhiraja, and R. Wu. Weakly interacting particle systems on inhomogeneous random graphs. *To appear in Stochastic Processes and their Applications*, arXiv:1612.00801.

[2] A. Budhiraja, D. Mukherjee and R. Wu. Supermarket model on graphs. *Accepted by The Annals of Applied Probability*, arXiv:1712.07607.

[3] S. Delattre, G. Giacomin, and E. Luçon. A note on dynamical models on random graphs and Fokker–Planck equations. *Journal of Statistical Physics*, 165(4):785–798, 2016.

[4] D. Lacker, K. Ramanan, and R. Wu. Local dynamics for large sparse networks of interacting Markov chains / diffusions. Near completion, 2018.

[5] A-S. Sznitman. Topics in propagation of chaos. *Ecole d'Eté de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Mathematics, vol 1464*, Springer, Berlin, Heidelberg, 1991.