Correction to Black-Scholes formula due to fractional stochastic volatility

Knut Sølna, UC Irvine

Collaborator:

Josselin Garnier, Paris VII

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↪ Motivation: Stochastic volatility modeling in finance.
↪ Market data suggests long range volatility correlations.
↪ Modeling with long range processes.
↪ Illustration equities and implied volatility surface.
↪ Mixing versus long range processes.
↪ Martingale method and price asymptotics.

Aspects and objectives:

◇ "Hidden" Volatility/Parameter Time Scales and parameter heterogeneity are important; leverage and clustering effects.
◇ Efficient and simple description of Stochastic Volatility effects using Perturbation Methods, under separartion of time scales and long range processes.
◇ Parsimonious "Effective Parameteric" representation for derivative Linkage and insight captured by perturbations.

Black-Scholes and Stochastic Volatility

• **Consider Black-Scholes model:**

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t \,,$$

with a stochastic volatility σ_t.

$\hookrightarrow$ **Quantity of interest: "operator perturbation" relative to constant volatility case:**

$$\mathcal{L}_{BS}(f(Y_t)) - \mathcal{L}_{BS}(\sigma) \;=\; \frac{1}{2}\left(\sigma_t^2 - \sigma^2\right) x^2 \partial_x^2 \,,$$

with

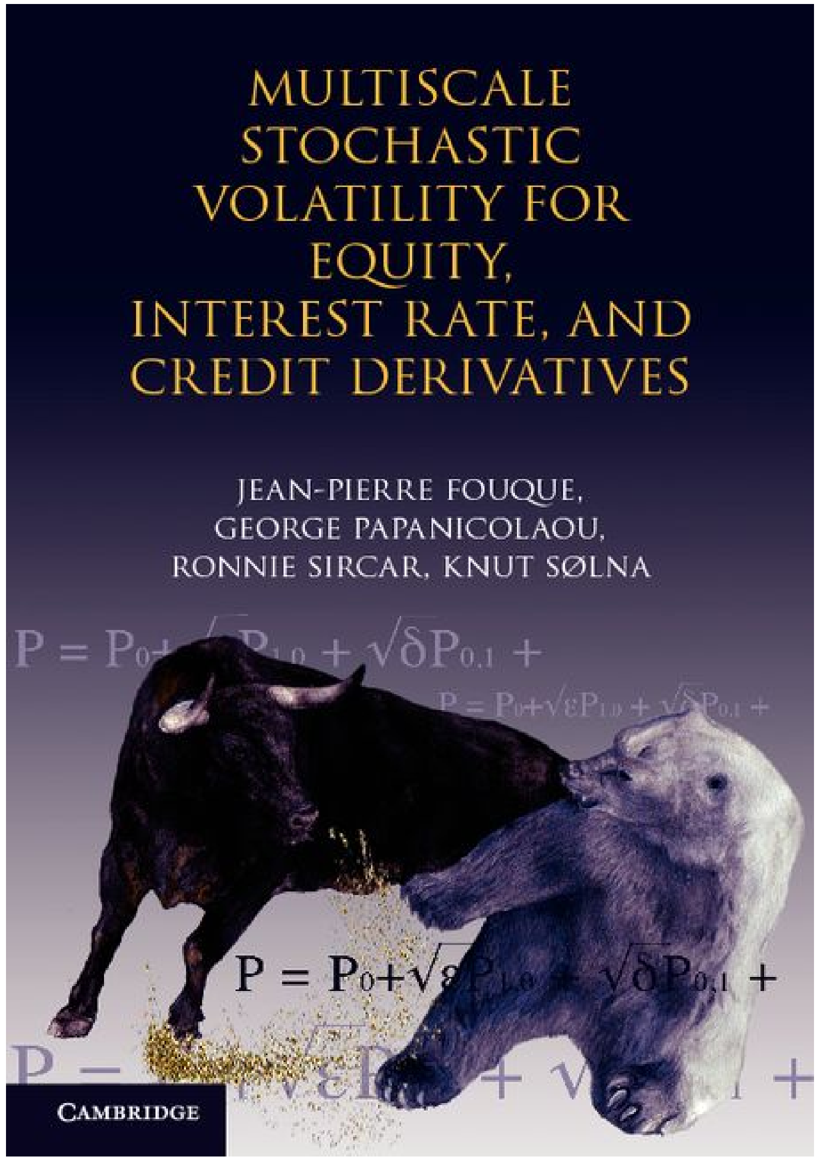
$$\mathcal{L}_{BS}(\sigma) \;=\; \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right).$$

$\hookrightarrow$ **Issue: Long memory in the volatility process.**

• **How should it be modeled**

& what is consequence for pricing, as measured by implied volatility.

“Mixing” case: σ_t Markov (diffusion):



Long-Range Correlation:

• **Long-range correlation/long memory process Z_t :**

"slow decorrelation".

We say that the random process Z_t has the H-long-range correlation property if its autocovariance function satisfies:

$$\mathcal{C}(\Delta t) = \mathbb{E}[Z_t Z_{t+\Delta t}] \overset{|\Delta t| \to \infty}{\simeq} r_H \left| a\Delta t \right|^{2H-2}, \tag{1}$$

where $r_H > 0$ and $H \in (1/2, 1)$. and refer to H as the Hurst exponent. Here the correlation time $1/a$ is the critical length scale beyond which the power law behavior (1) is valid.

$\rightarrow$ **Autocovariance function is not integrable, thus process not mixing.**

Short-Range Correlation:

- **Short-range correlation property:**

process is "rough at small scales".

Autocovariance function decays faster than an affine function at zero. The random process Z_t has the H-short-range correlation property if:

$$\mathcal{C}(\Delta t) = \mathbb{E}[Z_t Z_{t+\Delta t}] \overset{|\Delta t| \to 0}{\simeq} \mathcal{C}(0)\Big(1 - d_H \Big|a\Delta t\Big|^{2H} + O\Big(\Big|a\Delta t\Big|\Big)\Big), \tag{2}$$

where $d_H > 0$ and $H \in (0, 1/2)$. Here the correlation time $1/a$ is the critical length scale below which the power law behavior (2) is valid.

Integrable correlation function, but not a Markov process.

Model 0: Fractional Brownian motion:

A fractional Brownian motion (fBM) is a zero-mean Gaussian process $(W_t^H)_{t\in R}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

where σ_H is a positive constant. The fBm process, W_t^H, is also self-similar in that

$$\left\{W_{\alpha t}^H, t \in \boldsymbol{R}\right\} \stackrel{dist.}{=} \left\{\alpha^H W_t^H, t \in \boldsymbol{R}\right\} \text{ for all } \alpha > 0.$$

Stationary increments:

$$\mathbb{E}\left[(W^H(t) - W^H(s))^2\right] = |t-s|^{2H}$$

$H = 1/2$: **standard Brownian motion (independent increments).**
$H < 1/2$: **short-range correlations (negatively-correlated increments or "anti-persistent"). The realizations are continuous but very irregular.**
$H > 1/2$: **long-range correlations (positively-correlated increments). The realizations are continuous and more regular (but not differentiable).**

(sample paths W_H are a.s. Hölder continuous of order less than H).

However, the process itself is not stationary !

- **Moving-average stochastic integral representation of the fBM (Mandelbrot et al SIAM Review 1968):**

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_R (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW_s,$$

where $(W_t)_{t\in R}$ is a standard Brownian motion over $\boldsymbol{R}$.

Model 1: Fractional Ornstein Uhlenbeck (OU) process:

Define the a-scaled fractional Ornstein-Uhlenbeck process (fOU) as

$$Z_t^H = a^H \int_{-\infty}^t e^{-a(t-s)} dW_s^H = a^H W_t^H - a^{1+H} \int_{-\infty}^t e^{-a(t-s)} W_s^H ds.$$

where W_H is a fractional Brownian motion with Hurst index $H \in (0,1)$.

Looks like a fBm, but with a restoring force $\rightarrow$ stationary.

The moving-average integral representation of the fOU is then

$$Z_t^H = \int_{-\infty}^t \mathcal{K}(t-s) dW_s,$$

where

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \left[t^{H-\frac{1}{2}} - a \int_0^t (t-s)^{H-\frac{1}{2}} e^{-as} ds \right].$$

with in particular - $\mathcal{K}$ is nonnegative-valued, $\mathcal{K} \in L^2(0,\infty)$ for any $H \in (0,1)$.

Model 1: Fractional Ornstein Uhlenbeck Continued:

For $H \in (0, 1/2)$ the fOU process possesses short-range correlation properties:

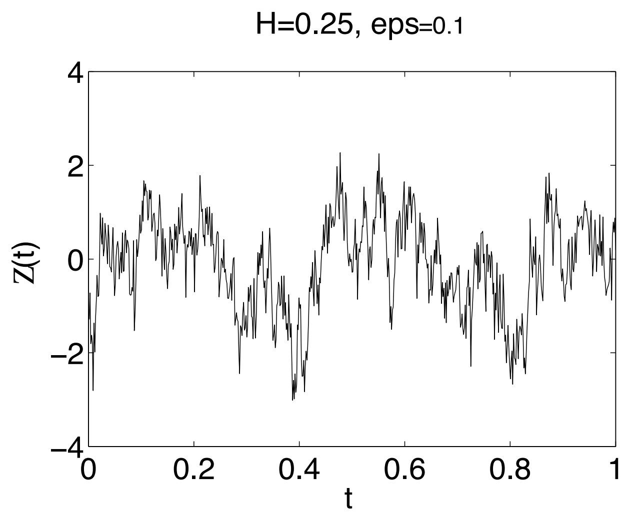
$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{\mathrm{ou}}^2 \Big(1 - \frac{1}{\Gamma(2H+1)}(as)^{2H} + o\big((as)^{2H}\big)\Big), \qquad as \ll 1.$$

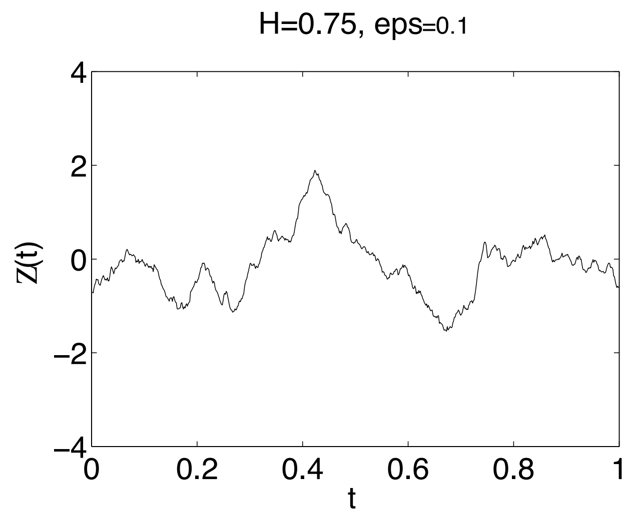
For $H \in (1/2, 1)$ it possesses long-range correlation properties:

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{\mathrm{ou}}^2 \Big(\frac{1}{\Gamma(2H-1)}(as)^{2H-2} + o\big((as)^{2H-2}\big)\Big), \qquad as \gg 1.$$

with $\sigma_{\mathrm{ou}}^2 = (2\sin(\pi H))^{-1}$.

Model 1: Fractional Ornstein Uhlenbeck Continued:





Realizations of the fractional Ornstein-Uhlenbeck process Z_t^H, with Hurst index H, and correlation length $a = l_c$. The trajectories are more regular when H is larger, $H = 1/2$ corresponds to a "standard" Ornstein-Uhlenbeck process. The process is a stationary random process, averaging with respect to its invariant distribution will be denoted by $< \cdot >$.

Ref: Cheridito et al, Electronic Journal of Prob 2003.

Stochastic Differential Equation for Risky Asset:

$$dX_t = \sigma_t X_t dW_t^*,$$

where the stochastic volatility is

$$\sigma_t = F(Z_t^H),$$

for Z_t^H and is adapted to W_t, and W_t^* is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1-\rho^2} B_t,$$

where the Brownian motion B_t is independent of W_t The function F is assumed to be one-to-one, positive valued, smooth, bounded and with bounded derivatives. Accordingly, the filtration $\mathcal{F}_t$ generated by (B_t, W_t) is also the one generated by X_t.

$\hookrightarrow$ The process σ_t is a stationary random process with mean $\mathbb{E}[\sigma_t] = \langle F \rangle$ and variance $\mathrm{Var}(\sigma_t) = \left\langle F^2 \right\rangle - \langle F \rangle^2$, independently of a, and "inherits" the short and long range correlation properties of the fractional Ornstein-Uhlenbeck process.

Remark: Model 2: Binary medium - long-range correlations

The process Z_t be stepwise constant.

$(l_j)_{j\geq 0}$**: lengths of the elementary intervals.**

$(n_j)_{j\geq 0}$**: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.**

The random variables n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The random variables l_j are i.i.d. with the pdf ($H \in (1/2, 1)$):

$$p_{l_1}(z) = \frac{3-2H}{l_c} \frac{l_c^{4-2H}}{z^{4-2H}} \mathbf{1}_{[l_c,\infty)}(z)$$

Note: The average length of the intervals is $\frac{3-2H}{2-2H} l_c$ while the variance is infinite. The process is (almost) stationary and we have for the covariance $C(z) \overset{|z|\to\infty}{\simeq} r_H \left|\frac{z}{l_c}\right|^{2H-2}$ with $r_H = \sigma^2/(3-2H)$.

Remark: Model 2: Binary medium - long-range correlations

The process ν is stepwise constant.
$(l_j)_{j\geq 0}$: lengths of the elementary intervals.
$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.
The random variables n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The random variables l_j are i.i.d. with the pdf ($H \in (0,1/2)$):

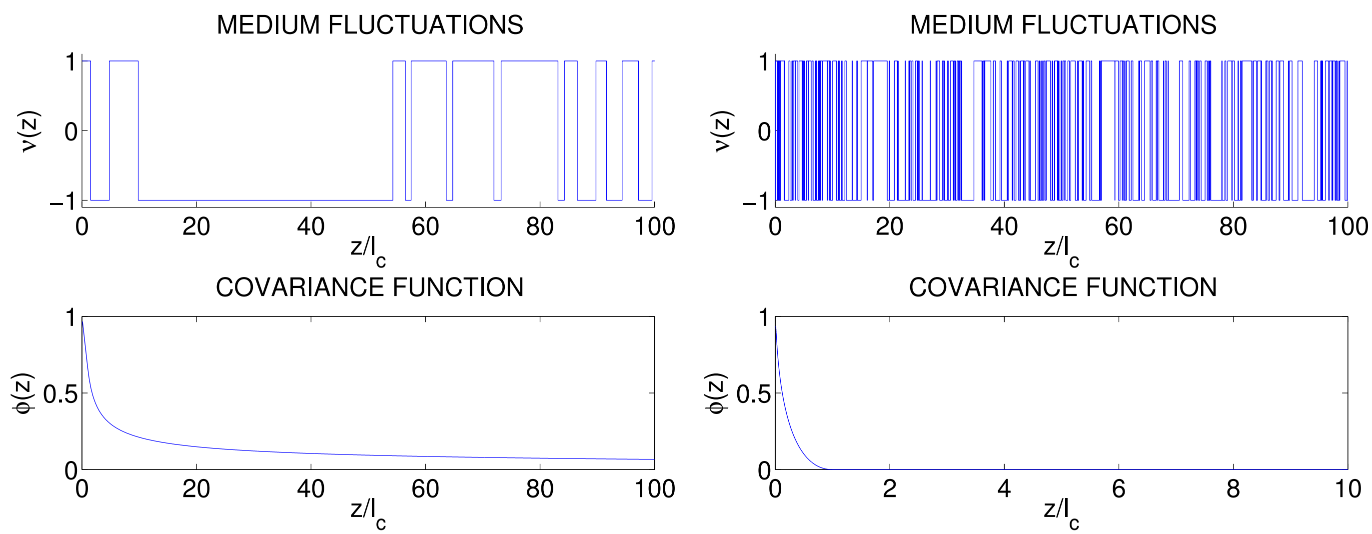
$$p_{l_1}(z) = \frac{1-2H}{l_c[(l_i/l_c)^{2H-1}-1]} \frac{l_c^{2-2H}}{z^{2-2H}} \mathbf{1}_{[l_i,l_c]}(z)$$

with $l_i \ll l_c$.

The process is (almost) stationary and we have for the covariance
We have $\phi(z) \overset{|z|\to 0}{\simeq} \sigma^2\big(1 - d_H\big|\frac{z}{l_c}\big|^{2H} + o\big(\big|\frac{z}{l_c}\big|\big)\big)$ with $d_H = \frac{1}{1-2H}$.

I: Illustration: Simple Binary medium with Short and Long Range Correlation

- **Long- or short-range correlations: slow or fast decay in $\mathcal{C}$:**



- **LEFT: Realization of a binary mediu, ν, with long-range correlation property, top plot. The "heavy-tailed" covariance function, $\mathcal{C}$, is in bottom plot.**
- **RIGHT: Same for short-range medium. This correlation function, $\mathcal{C}$, has then rapid decay at the origin. The short range property is due to the accumulation of small section lengths.**

Some References

• Gateral et al, arXiv 2014, Introduce "Rough Fractional Stochastic Volatility" $H < 1/2$ motivated by analysis of market data, use computational scheme for pricing. Modeling: Log-returns are fBm.

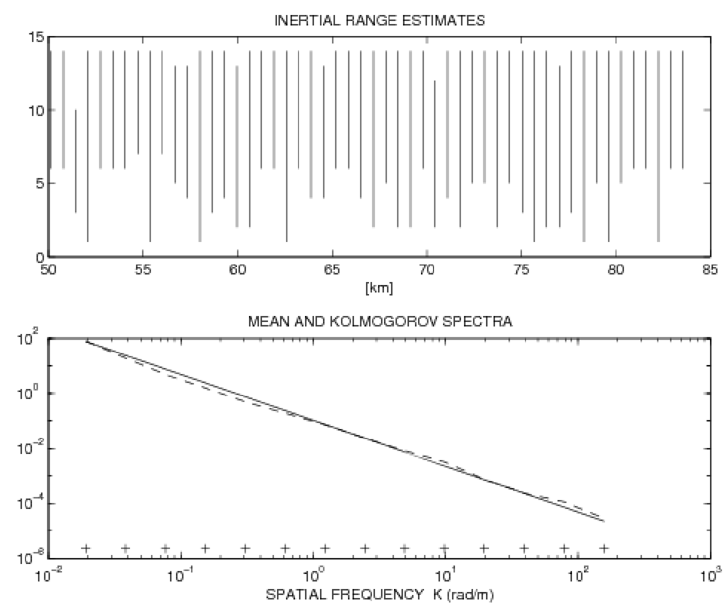
• Chronopoulou & Viens, Annals of Finance 2012, Estimate $H > 1/2$ use computational scheme for pricing. Modeling: Use smooth function of fOU model.

• Alòs et al Finance Stoch 2007. Use Malliavin calculus to get expressions for the implied volatility in the regime of small maturity.
Modeling: $\sigma_t = F(Y_t), \quad Y_t = c\int_0^t e^{-a(t-s)} dW_s^H, \quad W_s^H \equiv \int_0^s (s-u)^{H-1/2} dW_u$.

• Fukasawa Finance Stoch 2011. Discusses the case with small volatility fluctuations.
Modeling: For W_t as above $\sigma_t = f(m + \delta W_t)$.

• Mendes et al, Phys A 2015. Discuss in particular general well posedness, estimate $H > 1/2$.
Modeling: Volatiliy is function of fractional Gaussian noise.

Remark on Observations Power Law Media



Papanicolaou and S 2002.

Recall Set-Up

Derivative price: $P_t = \mathbb{E}^*[h(X_T) \mid \mathcal{F}_t]$ **(T-expiration).**

Call option: $h(x) = (x-K)^+$ **(K-strike), or with general payoff.**

Implied volatility I: $P^{\mathrm{BS}}(I) = P_{\mathrm{obs}}$.

- **Modeling:**

$$\begin{aligned} dX_t &= \sigma_t X_t dW_t^*, \quad \sigma_t = F(Z_t^H), \\ Z_t^H &= a^H \int_{-\infty}^t e^{-a(t-s)} dW_s^H = \int_{-\infty}^t \mathcal{K}(t-s) dW_s, \end{aligned}$$

special case $H = 1/2 : \mathcal{K}(s) = \sqrt{a}\exp(-as)$, **while for** $H \neq 1/2$ $\mathcal{K}$ **is an integral operator. Leverage Effect**: $\rho \neq 0$; **typically negative correlation between volatility and price fluctuations** $\rho dt = d < W, W^* >_t$.

$\hookrightarrow$ **Next on time scales and asymptotic pricing.**

Perturbations and Time Scales

• "Time scaling" of volatility factor "characteristic time scale" $1/a$ with time to maturity $T = O(1)$. Note, using notation $Z_t^a = Z_t \mid_a$:

$$Z_t^a \stackrel{\mathcal{D}}{=} Z^1_{t/a}.$$

$\hookrightarrow$ Three regimes (at least):

(i) $a \gg 1$, "fast mean reversion" , averaging and singular perturbation.

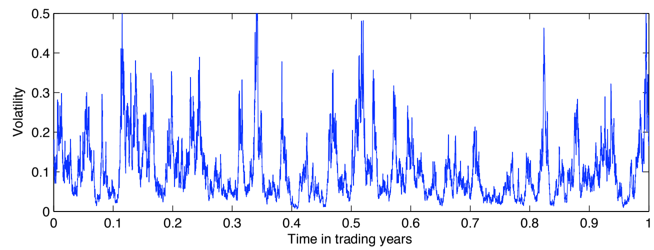
(ii) $a = O(1)$, "full interaction" , short time asymptotics feasible.

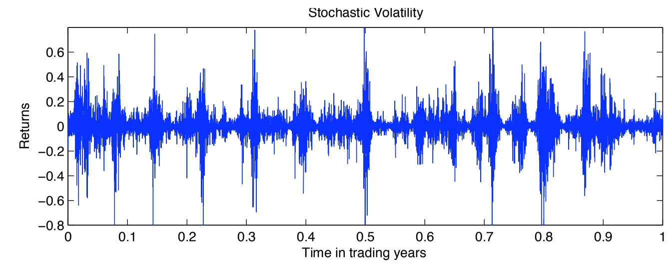
(iii) $a \ll 1$ & "slow dynamics" , regular perturbation.

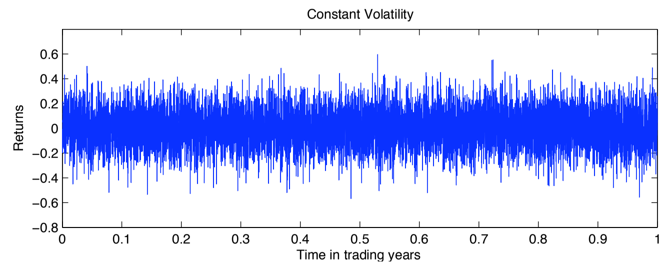
(iiiB) $a = 1$ & $Z_t^1 \mapsto \delta Z_t^1$ order one time scale, but small volatility fluctuations, regular perturbation.

$\hookrightarrow$ Remark: The fractional/ordinary Brownian motion contains "variations" on all scales, but with a "magnitude scaling" over the time cascade.

Example Fast Scale Volatility







• For illustration: “Fast” volatility process $\sigma_t = H(Z_t^a) = \exp(Z_t^a)$ with $a = 100$ and $H = 1/2$.

Top Left figure shows a realization of “bursty”, stochastic volatility σ_t for standard Ornstein-Uhlenbeck fast mean reverting volatility factor.
Top Right figure shows corresponding returns process.
Bottom figure shows returns with constant volatility.

Complicated Pricing problem

$\hookrightarrow$ **For $H \neq 1/2$ no pricing PDE.**

$\hookrightarrow$ **For $\rho \neq 0$ complicated pricing PDE via Feynman-Kac formula:**

$$\left\{ \underbrace{\frac{\partial}{\partial t} + \frac{1}{2} F^2(z) x^2 \frac{\partial^2}{\partial x^2} + r \left(x \frac{\partial}{\partial x} - \cdot \right)}_{\mathcal{L}_{BS}(F(z))} + \underbrace{\rho \sqrt{a} \left(x F(z) \frac{\partial^2}{\partial x \partial z} \right)}_{\textbf{Correlation}} + a \underbrace{\left(\frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} \right)}_{\mathcal{L}_Z} \right\} P = 0.$$

Remark: We do not consider market price of volatility risk here.

$\hookrightarrow$ **Consider first the mixing case with $H = 1/2$.**

Case (i): Fast Mean Reversion and Averaging

- **"Effective" volatility :**

$$\frac{1}{T}\int_0^T F^2(Z_t^a)\,dt \overset{a\to\infty}{=} \int F^2(z)\Phi_Z(z)dz = \left\langle F^2(z) \right\rangle \equiv \bar{\sigma}^2\,,$$

for Φ_Z the invariant distribution of Z_t^a.

- **"Accumulation" of perturbations and Poisson equation:**

$$\begin{aligned}
\mathcal{L}_{Z^1}\phi &= F^2(z) - \bar{\sigma}^2\,, && \text{Poisson equation} \\
\mathcal{L}_{Z^a} &= a\mathcal{L}_{Y^1}\,, && \text{generator scaling} \\
\hookrightarrow d\phi(Z_t^a) &= a\left(F^2(Z_t^a) - \bar{\sigma}^2\right) + \sqrt{a}\,\phi'(Z_t^a)dW_t^{(1/2)}\,.
\end{aligned}$$

$\hookrightarrow$ **Residual fluctuations (large but finite time scale separation):**

$$\int_0^T \left(F^2(Z_t^a) - \bar{\sigma}^2\right) dt = \frac{1}{a}\int_0^T \left(d\phi_t - \sqrt{a}\phi' dW_t^{(1/2)}\right) = O\left(\frac{1}{\sqrt{a}}\right)\,.$$

Remark $\mathbb{E}\left[\int_0^T \phi' dW_t^{(1/2)} \mid \mathcal{F}_0\right] = 0$.

Correction : Correlation with the BM driving the underlying gives correction at the order $1/\sqrt{a}$. Remark: Risk neutral: Market price of volatility risk gives correction at same order.

Case (ii): The “Contemporary” Scale $a = 1$

$\hookrightarrow$ **Details of model is important. Explicit results only available for (very) special, albeit practically important, models.**

Examples: $dX_t = \mu_t X_t dt + f(Y_t) X_t dW_t^x$.

- **Heston**:

$$\begin{aligned} f(y) &= \sqrt{y}, \\ dY_t &= \alpha(m - Y_t)dt + \beta\sqrt{Y_t}dW_t^y \ \ \text{CIR or Feller}, \\ d < W^y, W^x >_t &= \rho. \end{aligned}$$

Calibration over range of strikes and maturities challenging; computation via FFT.

- **Hull-White**:

$$\begin{aligned} f(y) &= \sqrt{y}, \\ dY_t &= \alpha Y_t dt + \beta Y_t dW_t^y \ \ \text{Lognormal}, \\ d < W^y, W^x >_t &= 0. \end{aligned}$$

Symmetric smiles; computation via conditioning on $\int_0^T f^2(Y_t)dt/T$.

Case (iii): The Slow Scale Modulation

- **Assume $z = Z_0^a$ the initial point :**

$$\frac{1}{T}\int_0^T F^2(Z_t^a)dt \xrightarrow{a\to 0} F^2(z).$$

Residual fluctuations:

$$\begin{aligned}
&\int_0^T \left(F^2(Z_t^a) - F^2(z)\right) dt \\
&= \int_0^T \left[F^2\left(z + a\int_0^t ds + \sqrt{a}\int_0^t dW_s\right) - F^2(z)\right] ds \\
&\sim 2F'(z)F(z)\left(\sqrt{a}\int_0^T W_t\, dt\right).
\end{aligned}$$

Remark again $\mathbb{E}\left[\int_0^T W_t\, dt \mid \mathcal{F}_0\right] = 0$.

Correction : Correlation with the BM driving the underlying gives correction at the order $\sqrt{a}$.

Case (iiiB): Small Volatility Fluctuations

- **Assume $z = Z_0^a$ the initial point :**

$$\frac{1}{T}\int_0^T F^2(\delta Z_t^1)\,dt \xrightarrow{\delta \to 0} F^2(0).$$

Residual fluctuations:

$$\int_0^T \left(F^2(\delta Z_t^1) - F^2(0)\right) dt$$
$$\sim 2F'(0)F(0)\left(\delta \int_0^T Z_t\,dt\right).$$

Correction : Correlation with the BM driving the underlying gives correction at the order δ.

Correction via Martingale Method

- **The price: $P_t = \mathbb{E}[h(X_T) \mid \mathcal{F}_t]$, a semi-martingale, here with $r = 0$ a martingale.**

- **Introduce the approximation:**

$$\tilde{P}_t = M_t + R_t$$

with Ansatz:
(i) $\tilde{P}_t = \tilde{P}(t, X_t)$, $\tilde{P}_T = h(x)$, (ii) M_t martingale, (iii) $R_t = O(\text{``}small'')$.

$\hookrightarrow$ **Then:**

$$\begin{aligned}
\tilde{P}_t &= M_t + R_t = \mathbb{E}[M_T \mid \mathcal{F}_t] + R_t \\
&= \mathbb{E}[M_T + R_T \mid \mathcal{F}_t] + R_t - \mathbb{E}[R_T \mid \mathcal{F}_t] \\
&= P_t + (R_t - \mathbb{E}[R_T \mid \mathcal{F}_t]) = P_t + O(\text{``}small'')\,.
\end{aligned}$$

Fast Scale Approximation in Mixing Case

$$\begin{aligned}
\tilde{P}_t - \tilde{P}_0 &\stackrel{\text{Ito}}{=} \int_0^t \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s\,ds + F(Z_s^a)X_s\partial_x\tilde{P}_s dW_s^* \\
&\quad + \frac{1}{2}\left(F^2(Z_s^a) - \bar{\sigma}^2\right) X_s^2\partial_x^2\tilde{P}_s\,ds \\
&\stackrel{\text{by Poisson}}{=} \int_0^t \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s\,ds + F(Z_s^a)X_s\partial_x\tilde{P}_s dW_s^{(x)} \\
&\quad + \frac{1}{2a}\left(d\phi - \sqrt{a}\phi'(Z_s^a)dW_s\right) X_s^2\partial_x^2\tilde{P}_s \\
&\stackrel{\text{by parts}}{=} \int_0^t \left(\underline{\mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s - \sqrt{1/a}\,\mathcal{V}(Z_s^a)(X_s\partial_x)(X_s^2\partial_x^2)\tilde{P}_s}\right) ds
\end{aligned}$$

$$+..dW_s^* + ..dW_s + O(1/a) \stackrel{\text{objective}}{=} \text{martingale} + O(1/a)\,.$$

$$\begin{aligned}
d\phi X_s^2\partial_x^2\tilde{P}_s &= d\left(\phi X_s^2\partial_x^2\tilde{P}_s\right) - \phi d\left(X_s^2\partial_x^2\tilde{P}_s\right) - d\left\langle \phi, x^2\partial_x^2\tilde{P}\right\rangle_s, \\
d\left\langle \phi, x^2\partial_x^2\tilde{P}\right\rangle_s &= \sqrt{a}\rho\phi'(Z_s^a)F(Z_s^a)(X_s\partial_x)(X_s^2\partial_x^2)\tilde{P}_s ds \\
&\equiv \sqrt{a}\,\mathcal{V}(Z_s^a)(X_s\partial_x)(X_s^2\partial_x^2)\tilde{P}_s\,ds\,. \\
&\hookrightarrow \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_t - 1/\sqrt{a}\left\langle \mathcal{V}(Z_t)\right\rangle_{\Phi_Z}(x\partial_x)(x^2\partial_x^2)\tilde{P}_t = 0\,; \tilde{P}_T = h\,.
\end{aligned}$$

Recall key step in mixing argument, the Poisson step:

$$\mathcal{L}_{Z^1}\phi \;=\; F^2(z) - \bar{\sigma}^2\,, \quad \hookrightarrow \left(F^2(Z_t^a) - \bar{\sigma}^2\right) = \frac{1}{a}\left(d\phi(Z_t^a) - \sqrt{a}\,\phi'(Z_t^a)dW_t^{(1/2)}\right).$$

Long range case is much harder!! Useful quantity: the conditionally expected volatility fluctuations over remaining time epoch:

$$\phi_t^a = \mathbb{E}\Big[\frac{1}{2}\int_t^T \left((\sigma_s^a)^2 - \overline{\sigma}^2\right)ds \Big| \mathcal{F}_t\Big] = O(\text{“}small''\text{)}$$

Decompose as

$$\phi_t^a = \psi_t^a - \frac{1}{2}\int_0^t \left(F(Z_s^a)^2 - \overline{\sigma}^2\right)ds,$$

where the martingale ψ_t^a is defined by

$$\psi_t^a = \mathbb{E}\Big[\frac{1}{2}\int_0^T \left(F(Z_s^a)^2 - \overline{\sigma}^2\right)ds \Big| \mathcal{F}_t\Big],$$

so that we have

$$\left(F^2(Z_t^a) - \bar{\sigma}^2\right) = 2\left(d\psi(Z_t^a) - d\phi(Z_t^a)\right).$$

Fast Long Range Case; Continued

$\hookrightarrow$ **To identify problem for the correction we also need, as above, covariation with asset. We have:**

$$d\langle \psi^a, W\rangle_t = \left(a^{H-1}\theta_t + \widetilde{\theta}^a_t\right)dt,$$

for

$$\theta_t = \ (T-t)^{H-\frac{1}{2}}\frac{\sigma_{\rm ou}\left\langle FF'\right\rangle}{\Gamma(H+\frac{3}{2})},$$

and $\widetilde{\theta}^a_t$ is random but small:

$$\limsup_{a\to\infty} a^{1-H} \sup_{t\in[0,T]} \mathbb{E}\left[(\widetilde{\theta}^a_t)^2\right]^{1/2} = 0.$$

$\rightarrow$ **Rapid volatility fluctuations reflects themselves in a small covariation.**

Fast Long Range Case; Continued

To deal with a general (smooth) function of the volatility fluctuations and get bounds for these we use a Hermite decomposition for the stochastic volatility We denote

$$\widetilde{F}(z) = F(\sigma_{\mathrm{ou}} z)^2.$$

Because $\mathbb{E}[\widetilde{F}(Z)^2] < \infty$ is finite when Z is a standard normal variable, the function $\widetilde{F}$ can be expanded in terms of the Hermite polynomials

$$H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2}$$

and the series

$$\sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(z),$$

$$C_k = \mathbb{E}\left[H_k(Z)\widetilde{F}(Z)\right] = \int_R H_k(z)\widetilde{F}(z)p(z)dz,$$

converges in $L^2(\boldsymbol{R}, p(z)dz)$ to $\widetilde{F}(z)$.

Price Approximation Fast Long Range Case:

$$P_t = Q_t^a(X_t) + o(a^{H-1}),$$

where

$$Q_t^a(x) = Q_t^{(0)}(x) + \left(x^2\partial_x^2 Q_t^{(0)}(x)\right)\phi_t^a + a^{H-1}\rho Q_t^{(1)}(x),$$

with

$$\mathcal{L}_{\mathrm{BS}}(\overline{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x),$$

and ϕ_t^ε is the random path dependent component

$$\phi_t^\varepsilon = \mathbb{E}\Big[\frac{1}{2}\int_t^T \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right)ds \big| \mathcal{F}_t\Big],$$

moreover, $Q_t^{(1)}(x)$ is the deterministic correction

$$Q_t^{(1)}(x) = x\partial_x\left(x^2\partial_x^2 Q_t^{(0)}(x)\right)\left[(T-t)^{H+\frac{1}{2}}\frac{\sigma_{\mathrm{ou}}\langle F\rangle\langle FF'\rangle}{\Gamma(H+\frac{3}{2})}\right].$$

- **Case $H = 1/2$ gives deterministic ϕ_t^ε and reordering of terms, but we have "continuity" with respect to qualitative form.**

Representation of Price Correction Fast Long Range Call Case

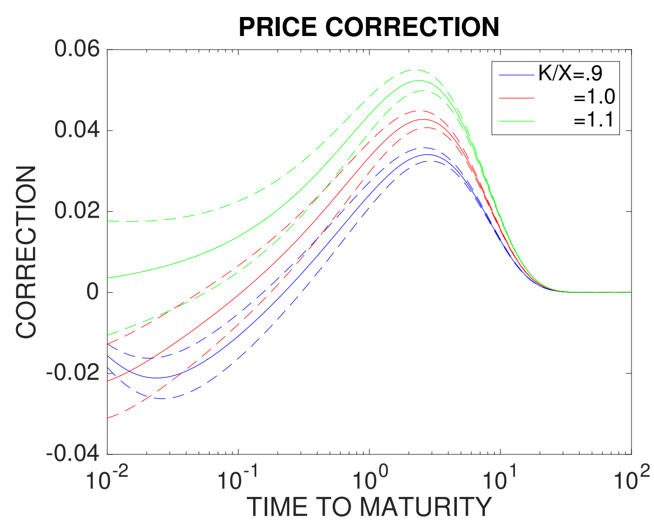
$$\frac{1}{K}\left(\phi_t^\varepsilon\left(x^2\partial_x^2\right)Q_t^{(0)}(x)+a^{H-1}\rho Q_t^{(1)}(x)\right)$$
$$=\left(\frac{e^{-d_1^2/2}\frac{x}{K}}{\sqrt{\pi}}\right)\left\{\frac{\phi_t^\varepsilon}{2}\left(\frac{\tau}{\bar{\tau}}\right)^{-1/2}+a_F\left[\left(\frac{\tau}{\bar{\tau}}\right)^{H}+\left(\frac{\tau}{\bar{\tau}}\right)^{H-1}\log\frac{K}{x}\right]\right\},$$

with $\tau=T-t$ **and the effective diffusion time** $\bar{\tau}=2/\bar{\sigma}^2$**, moreover with:**

$$\begin{aligned}
d_1 &= \frac{\frac{\tau}{\bar{\tau}}-\log\frac{K}{x}}{\sqrt{2\frac{\tau}{\bar{\tau}}}},\\
a_F &= a^{H-1}\frac{\rho\sigma_{\mathrm{ou}}\langle F\rangle\langle FF'\rangle\bar{\tau}^H}{2^{3/2}\overline{\sigma}\Gamma(H+\frac{3}{2})},\\
\phi_t^\varepsilon &= O\left(\left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}\left(\frac{\tau}{\bar{\tau}}\right)^{H}\right).
\end{aligned}$$

Price Correction as Function of Relative Maturity

The figure shows the relative price correction as function of relative time to maturity $\tau/\bar{\tau}$ for 3 values of the moneyness K/x. The solid lines is the mean price and the dashed lines gives the mean price plus/minus one standard deviation. We used here $a_F = 0.1$ and $\left((1/(a\bar{\tau})^{(1-H)})\bar{\tau}\sigma_\phi\right)^{1/2} = 0.02$.

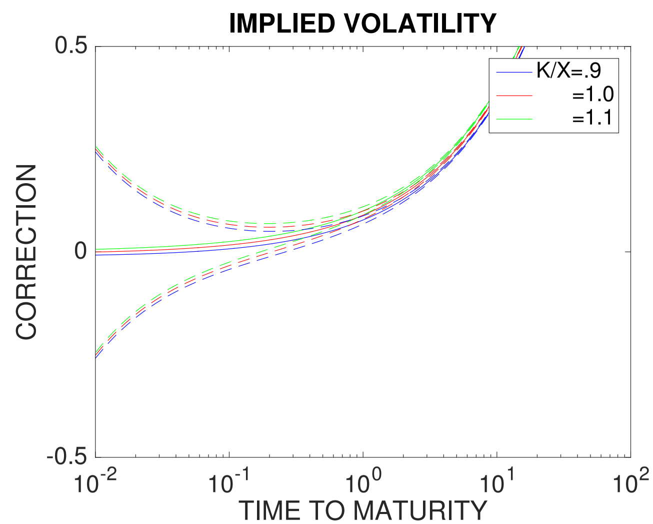


Implied Volatility Fast Long Range Call Case

$$
\begin{aligned}
I_t &= \overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} + \overline{\sigma} a_F \left[\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\bar{\tau}}\right)^{H-3/2} \log\left(\frac{K}{X_t}\right) \right] + o(\varepsilon^{1-H}) \\
&= \mathbb{E}\Big[\frac{1}{T-t} \int_t^T (\sigma_s^\varepsilon)^2 ds \big| \mathcal{F}_t \Big]^{1/2} + \overline{\sigma} a_F \left[\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\bar{\tau}}\right)^{H-3/2} \log\left(\frac{K}{X_t}\right) \right] + o(\varepsilon^{1-H}).
\end{aligned}
$$

$\rightarrow$ **We get fractional correction to expected volatility fluctuations, moreover, the correction diverges in the long maturity limit when the volatility fluctuations have correlations that decays slowly (as a fractional power).**

The figure shows the implied volatility correction as function of relative time to maturity $\tau/\bar{\tau}$ for 3 values of the moneyness K/x. The solid lines is the mean correction and the dashed lines gives the mean correction plus/minus one standard deviation, parameters as above.



Statistical Structure of Surface Fluctuations

As $a \to \infty$, the random process $a^{1-H}\phi^a_{t,T}$, $t \le T$, converges in distribution (in the sense of finite-dimensional distributions) to a centered Gaussian random process $\phi_{t,T}$, $t \le T$, we have the general covariance covariance function of $\phi^a_{t,T}$ has the following limit for any $t \le T$, $t' \le T'$, with $t \le t'$:

$$a^{2-2H}\mathbb{E}[\phi^a_{t,T}\phi^a_{t',T'}] \stackrel{a\to\infty}{\longrightarrow} \sigma^2_\phi (T-t)^H (T'-t')^H C_\phi(t,t';T,T'),$$

with $\sigma^2_\phi = \sigma^2_{\mathrm{ou}} \langle FF' \rangle^2 \left(1/\Gamma(2H+1)\sin(\pi H) - 1/2H\Gamma(H+\frac{1}{2})^2 \right)$,
and where the limit correlation is

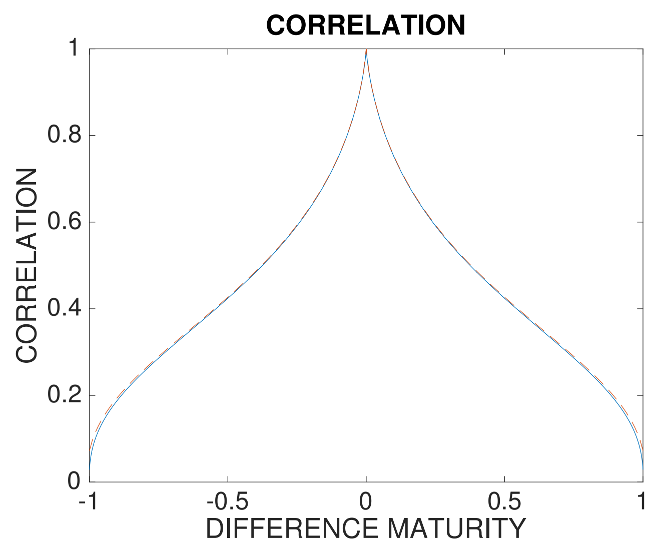
$$C_\phi(t,t';T,T') = \frac{\int_0^\infty du \big[(u+r)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]\big[(u+s)^{H-\frac{1}{2}} - (u+q)^{H-\frac{1}{2}}\big]}{\int_0^\infty du \big[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]^2},$$

with

$$q = \frac{t'-t}{\sqrt{(T-t)(T'-t')}}, \qquad r = \frac{\sqrt{T-t}}{\sqrt{T'-t'}}, \qquad s = \frac{T'-t}{\sqrt{(T-t)(T'-t')}}.$$

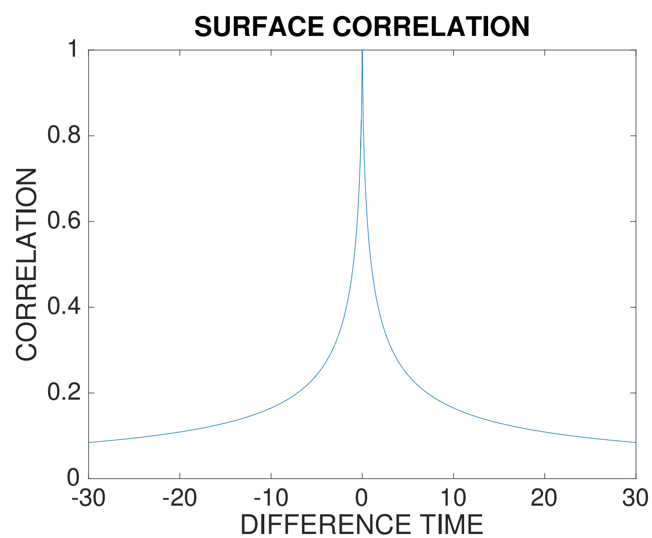
Canonical Decorrelation of Correction Coefficient

The figure shows how the stochastic correction coefficient decorrelate. For $T = T'$ we show $\mathcal{C}_\phi(t,t',T,T')$ as function of $|\tau - \tau'|/|\tau + \tau'|$.



Flapping of the Implied Surface

The figure shows how the stochastic correction coefficient decorrelate with respect to time translation. We consider the case with $T-t=T'-t'$ and plot the correlation coefficient as function of $|t'-t|/T-t|$. Note the rapid decay at the origin followed by a long range behavior. This shows how the price correction surface decorrelate as we move in time.



Implied Volatility Small Fractional Volatility Fluctuations:

$$I_t = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T F(Z_s^a)^2 ds \,|\, \mathcal{F}_t\Big]^{\frac{1}{2}} + \mathcal{A}(T-t)\Big[1 + \frac{\log(K/X_t)}{(T-t)/\bar{\tau}}\Big],$$

for

$$\mathcal{A}(\tau) = \frac{\delta\rho\bar{\sigma}\tau^{H+\frac{1}{2}}}{2\Gamma(H+\frac{5}{2})}\Big\{1 - \int_0^{a\tau} e^{-v}\big(1 - \frac{v}{a\tau}\big)^{H+\frac{3}{2}} dv\Big\},$$

In the short and long time to maturity regimes we then have for the leverage term:

$$\mathcal{A}(\tau)\Big[1 + \frac{\log(K/X_t)}{\tau/\bar{\tau}}\Big] = \begin{cases} a_{\mathrm{s}}\Big[(\tau/\bar{\tau})^{\frac{1}{2}+H} + (\tau/\bar{\tau})^{-\frac{1}{2}+H}\log(K/X_t)\Big] & \textbf{for } a\tau \ll 1, \\ a_1\Big[(\tau/\bar{\tau})^{-\frac{1}{2}+H} + (\tau/\bar{\tau})^{-\frac{3}{2}+H}\log(K/X_t)\Big] & \textbf{for } a\tau \gg 1, \end{cases}$$

Remark on Accuracy Call Case

Note that in the call case there is a singular payoff function.

One has to show that $\mathbb{E}\left[R_T - R_t \middle| \mathcal{F}_t\right]$ is "small" which involves bounds on higher order derivatives of the Black-Schloes price and showing that the local martingale introduced in the price approximation derivation indeed are martingales (up to time T).

Final Remarks

- Have analyzed fractional stochastic volatility.

- Analysis very different from (pde approach in) mixing case, main tool used is Martingale Method.

- Framework general.

- Long range correlation gives a random implied volatility surface with fractional term structure.

- Calibration parameters provide linkage between financial products.