Correction to Black-Scholes formula due to fractional stochastic volatility

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Motivation: Stochastic volatility modeling in finance.
Market data suggests long range volatility correlations.
Modeling with long range processes.
Illustration equities and implied volatility surface.
Mixing versus long range processes.
Martingale method and price asymptotics.

Aspects and objectives:

- “Hidden” Volatility/Parameter Time Scales and parameter heterogeneity are important; leverage and clustering effects.
- Efficient and simple description of Stochastic Volatility effects using Perturbation Methods, under separation of time scales and long range processes.
- Parsimonious “Effective Parameteric” representation for derivative Linkage and insight captured by perturbations.
• Consider Black-Scholes model:

\[ dX_t = \mu X_t dt + \sigma_t X_t dW_t, \]

with a stochastic volatility \( \sigma_t \).

\( \hookrightarrow \) **Quantity of interest:** “**operator perturbation**” relative to constant volatility case:

\[
\mathcal{L}_{BS}(f(Y_t)) - \mathcal{L}_{BS}(\sigma) = \frac{1}{2} \left( \sigma_t^2 - \sigma^2 \right) x^2 \partial^2_x,
\]

with

\[
\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2} \sigma_t^2 x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right).
\]

\( \hookrightarrow \) **Issue:** Long memory in the volatility process.

• How should it be modeled

& what is consequence for pricing, as measured by implied volatility.
“Mixing” case: $\sigma_t$ Markov (diffusion):
Long-Range Correlation:

- **Long-range correlation/long memory process** $Z_t$:
  “slow decorrelation”.

We say that the random process $Z_t$ has the $H$-long-range correlation property if its autocovariance function satisfies:

$$C(\Delta t) = \mathbb{E}[Z_tZ_{t+\Delta t}] \xrightarrow{\Delta t \to \infty} r_H \left|a \Delta t\right|^{2H-2},$$

where $r_H > 0$ and $H \in (1/2, 1)$. and refer to $H$ as the **Hurst exponent**. Here the correlation time $1/a$ is the critical length scale beyond which the power law behavior (1) is valid.

→ Autocovariance function is **not integrable**, thus process not mixing.
Short-Range Correlation:

- **Short-range correlation property:**
  process is “rough at small scales”.

  Autocovariance function decays faster than an affine function at zero. The random process $Z_t$ has the $H$-short-range correlation property if:

  \[
  C(\Delta t) = \mathbb{E}[Z_t Z_{t+\Delta t}] \mid_{\Delta t \to 0} \approx C(0) \left( 1 - d_H |a\Delta t|^{2H} + O\left( |a\Delta t| \right) \right),
  \]

  (2)

  where $d_H > 0$ and $H \in (0, 1/2)$. Here the correlation time $1/a$ is the critical length scale below which the power law behavior (2) is valid.

  Integrable correlation function, but not a Markov process.
Model 0: Fractional Brownian motion:

A fractional Brownian motion (fBM) is a zero-mean Gaussian process \((W^H_t)_{t \in \mathbb{R}}\) with the covariance

\[
\mathbb{E}[W^H_t W^H_s] = \frac{\sigma_H^2}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}),
\]

where \(\sigma_H\) is a positive constant. The fBM process, \(W^H_t\), is also self-similar in that

\[
\{W^H_{\alpha t}, t \in \mathbb{R}\} \overset{dist.}{=} \{\alpha^H W^H_t, t \in \mathbb{R}\} \text{ for all } \alpha > 0.
\]

Stationary increments:

\[
\mathbb{E}[(W^H(t) - W^H(s))^2] = |t - s|^{2H}
\]

- \(H = 1/2\): standard Brownian motion (independent increments).
- \(H < 1/2\): short-range correlations (negatively-correlated increments or “anti-persistent”). The realizations are continuous but very irregular.
- \(H > 1/2\): long-range correlations (positively-correlated increments). The realizations are continuous and more regular (but not differentiable).

(sample paths \(W_H\) are a.s. Hölder continuous of order less than \(H\)).

However, the process itself is not stationary!

- Moving-average stochastic integral representation of the fBM (Mandelbrot et al SIAM Review 1968):

\[
W^H_t = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} (t - s)^{H - \frac{1}{2}} - (-s)^{H - \frac{1}{2}} dW_s,
\]

where \((W_t)_{t \in \mathbb{R}}\) is a standard Brownian motion over \(\mathbb{R}\).
Define the \( a \)-scaled fractional Ornstein-Uhlenbeck process (fOU) as

\[
Z^H_t = a^H \int_{-\infty}^t e^{-a(t-s)} dW^H_s = a^H W^H_t - a^{1+H} \int_{-\infty}^t e^{-a(t-s)} W^H_s \, ds.
\]

where \( W_H \) is a fractional Brownian motion with Hurst index \( H \in (0, 1) \).

Looks like a fBm, but with a restoring force \( \rightarrow \) stationary.

The moving-average integral representation of the fOU is then

\[
Z^H_t = \int_{-\infty}^t K(t-s) dW_s,
\]

where

\[
K(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \left[ t^{H-\frac{1}{2}} - a \int_0^t (t-s)^{H-\frac{1}{2}} e^{-as} \, ds \right].
\]

with in particular - \( K \) is nonnegative-valued, \( K \in L^2(0, \infty) \) for any \( H \in (0, 1) \).
For $H \in (0, 1/2)$ the fOU process possesses short-range correlation properties:

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{\text{ou}}^2 \left( 1 - \frac{1}{\Gamma(2H+1)} (as)^{2H} + o((as)^{2H}) \right), \quad as \ll 1.$$ 

For $H \in (1/2, 1)$ it possesses long-range correlation properties:

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{\text{ou}}^2 \left( \frac{1}{\Gamma(2H-1)} (as)^{2H-2} + o((as)^{2H-2}) \right), \quad as \gg 1.$$ 

with $\sigma_{\text{ou}}^2 = (2 \sin(\pi H))^{-1}$. 

Realizations of the fractional Ornstein-Uhlenbeck process $Z_t^H$, with Hurst index $H$, and correlation length $a = l_c$. The trajectories are more regular when $H$ is larger, $H = 1/2$ corresponds to a “standard” Ornstein-Uhlenbeck process. The process is a stationary random process, averaging with respect to its invariant distribution will be denoted by $< \cdot >$.

Stochastic Differential Equation for Risky Asset:

\[ dX_t = \sigma_t X_t dW_t^*, \]

where the stochastic volatility is

\[ \sigma_t = F(Z_t^H), \]

for \( Z_t^H \) and is adapted to \( W_t \), and \( W_t^* \) is a Brownian motion that is correlated to the stochastic volatility through

\[ W_t^* = \rho W_t + \sqrt{1 - \rho^2} B_t, \]

where the Brownian motion \( B_t \) is independent of \( W_t \) The function \( F \) is assumed to be one-to-one, positive valued, smooth, bounded and with bounded derivatives. Accordingly, the filtration \( \mathcal{F}_t \) generated by \( (B_t, W_t) \) is also the one generated by \( X_t \).

\[ \text{The process } \sigma_t \text{ is a stationary random process with mean } \mathbb{E}[\sigma_t] = \langle F \rangle \text{ and variance } \text{Var}(\sigma_t) = \langle F^2 \rangle - \langle F \rangle^2, \text{ independently of } a, \text{ and "inherits" the short and long range correlation properties of the fractional Ornstein-Uhlenbeck process.} \]
Remark: Model 2: Binary medium - long-range correlations

The process $Z_t$ be stepwise constant.

$(l_j)_{j \geq 0}$: lengths of the elementary intervals.

$(n_j)_{j \geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The random variables $n_j$ are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm \sigma) = \frac{1}{2}$$

The random variables $l_j$ are i.i.d. with the pdf $(H \in (1/2, 1))$:

$$p_{l_1}(z) = \frac{3 - 2H}{l_c} \frac{l_c^{4-2H}}{z^{4-2H}} 1_{[l_c, \infty)}(z)$$

Note: The average length of the intervals is $\frac{3 - 2H}{2 - 2H} l_c$ while the variance is infinite. The process is (almost) stationary and we have for the covariance

$$C(z) \xrightarrow{|z| \to \infty} r_H \left| \frac{z}{l_c} \right|^{2H-2} \text{ with } r_H = \frac{\sigma^2}{(3 - 2H)}.$$
Remark: Model 2: Binary medium - long-range correlations

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The random variables $n_j$ are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm \sigma) = \frac{1}{2}$$

The random variables $l_j$ are i.i.d. with the pdf ($H \in (0, 1/2)$):

$$p_{l_1}(z) = \frac{1 - 2H}{l_c[(l_i/l_c)^{2H} - 1]} \frac{l_c^{2-2H}}{z^{2-2H}} 1[l_i,l_c](z)$$

with $l_i \ll l_c$.

The process is (almost) stationary and we have for the covariance

We have $\phi(z) \mid_{z \to 0} \approx \sigma^2 \left(1 - d_H \mid \frac{z}{l_c} \mid^{2H} + o\left(\mid \frac{z}{l_c} \mid \right)\right)$ with $d_H = \frac{1}{1-2H}$.
Long- or short-range correlations: slow or fast decay in $C$:

**LEFT:** Realization of a binary medium, $\nu$, with long-range correlation property, top plot. The “heavy-tailed” covariance function, $C$, is in bottom plot.

**RIGHT:** Same for short-range medium. This correlation function, $C$, has then rapid decay at the origin. The short range property is due to the accumulation of small section lengths.
Some References

- Gateral et al, arXiv 2014, Introduce “Rough Fractional Stochastic Volatility” $H < 1/2$ motivated by analysis of market data, use computational scheme for pricing. Modeling: Log-returns are fBm.


- Alòs et al Finance Stoch 2007. Use Malliavin calculus to get expressions for the implied volatility in the regime of small maturity. 
  **Modeling:** $\sigma_t = F(Y_t)$, $Y_t = c \int_0^t e^{-a(t-s)} dW_s^H$, $W_s^H \equiv \int_0^s (s-u)^{H-1/2} dW_u$.

  **Modeling:** For $W_t$ as above $\sigma_t = f(m + \delta W_t)$.

- Mendes et al, Phys A 2015. Discuss in particular general well posedness, estimate $H > 1/2$. 
  **Modeling:** Volatility is function of fractional Gaussian noise.
Remark on Observations Power Law Media

Papanicolaou and S 2002.
**Recall Set-Up**

**Derivative price:** \( P_t = E^* [h(X_T) \mid \mathcal{F}_t] (T\text{-expiration}). \)

**Call option:** \( h(x) = (x - K)^+ (K\text{-strike}), \) or with general payoff.

**Implied volatility I:** \( P_{BS}(I) = P_{obs}. \)

- **Modeling:**

\[
\begin{align*}
    dX_t &= \sigma_t X_t dW_t^*, \quad \sigma_t = F(Z_t^H), \\
    Z_t^H &= a^H \int_t^\infty e^{-a(t-s)} dW_s^H = \int_t^- \mathcal{K}(t-s) dW_s,
\end{align*}
\]

special case \( H = 1/2 : \mathcal{K}(s) = \sqrt{a} \exp(-as), \) while for \( H \neq 1/2 \) \( \mathcal{K} \) is an integral operator. **Leverage Effect:** \( \rho \neq 0; \) typically negative correlation between volatility and price fluctuations \( \rho dt = d < W, W^* >_t. \)

\( \leftrightarrow \) **Next on time scales and asymptotic pricing.**
• “Time scaling” of volatility factor “characteristic time scale” \( 1/a \) with time to maturity \( T = O(1) \). Note, using notation \( Z^a_t = Z_t \mid a \):

\[
Z^a_t \overset{\text{D}}{=} Z^1_t / a.
\]

\[\mapsto\] Three regimes (at least):

(i) \( a \gg 1 \), “fast mean reversion”, averaging and singular perturbation.

(ii) \( a = O(1) \), “full interaction”, short time asymptotics feasible.

(iii) \( a \ll 1 \& \) “slow dynamics”, regular perturbation.

(iiiB) \( a = 1 \& Z^1_t \leftrightarrow \delta Z^1_t \) order one time scale, but small volatility fluctuations, regular perturbation.

\[\mapsto\] Remark: The fractional/ordinary Brownian motion contains “variations” on all scales, but with a “magnitude scaling” over the time cascade.
For illustration: “Fast” volatility process $\sigma_t = H(Z_t^a) = \exp(Z_t^a)$ with $a = 100$ and $H = 1/2$.

Top Left figure shows a realization of “bursty”, stochastic volatility $\sigma_t$ for standard Ornstein-Uhlenbeck fast mean reverting volatility factor. Top Right figure shows corresponding returns process. Bottom figure shows returns with constant volatility.
Complicated Pricing problem

For $H \neq 1/2$ no pricing PDE.

For $\rho \neq 0$ complicated pricing PDE via Feynman-Kac formula:

$$\left\{ \frac{\partial}{\partial t} + \frac{1}{2} F^2(z) x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} \cdot \right) + \rho \sqrt{a} \left( x F(z) \frac{\partial^2}{\partial x \partial z} \right) + a \left( \frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} \right) \right\} P = 0.$$

Correlation

Remark: We do not consider market price of volatility risk here.

Consider first the mixing case with $H = 1/2$. 
Case (i): Fast Mean Reversion and Averaging

- “Effective” volatility:

\[
\frac{1}{T} \int_0^T F^2(Z_t^a) \, dt \xrightarrow{a \to \infty} \int F^2(z) \Phi_Z(z) \, dz = \left\langle F^2(z) \right\rangle \equiv \bar{\sigma}^2,
\]

for \( \Phi_Z \) the invariant distribution of \( Z_t^a \).

- “Accumulation” of perturbations and Poisson equation:

\[
\mathcal{L}_{Z^1} \phi = F^2(z) - \bar{\sigma}^2, \quad \text{Poisson equation}
\]

\[
\mathcal{L}_{Z^a} = a \mathcal{L}_{Y^1}, \quad \text{generator scaling}
\]

\[
\leftrightarrow d\phi(Z_t^a) = a \left( F^2(Z_t^a) - \bar{\sigma}^2 \right) + \sqrt{a} \phi'(Z_t^a) dW_t^{(1/2)}.
\]

\[\leftrightarrow \text{Residual fluctuations (large but finite time scale separation)}:
\]

\[
\int_0^T \left( F^2(Z_t^a) - \bar{\sigma}^2 \right) \, dt = \frac{1}{a} \int_0^T \left( d\phi_t - \sqrt{a} \phi' \, dW_t^{(1/2)} \right) = O \left( \frac{1}{\sqrt{a}} \right).
\]

Remark \( \mathbb{E} \left[ \int_0^T \phi' \, dW_t^{(1/2)} \mid \mathcal{F}_0 \right] = 0. \)

Correction: Correlation with the BM driving the underlying gives correction at the order \( 1/\sqrt{a} \). Remark: Risk neutral: Market price of volatility risk gives correction at same order.
Case (ii): The “Contemporary” Scale $a = 1$

Details of model is important. Explicit results only available for (very) special, albeit practically important, models.

Examples: $dX_t = \mu_t X_t dt + f(Y_t) X_t dW_t^x$.

- **Heston**:

\[
\begin{align*}
    f(y) &= \sqrt{y}, \\
    dY_t &= \alpha(m - Y_t)dt + \beta \sqrt{Y_t} dW_t^y \quad \text{CIR or Feller,} \\
    d < W^y, W^x >_t &= \rho.
\end{align*}
\]

Calibration over range of strikes and maturities challenging; computation via FFT.

- **Hull-White**:

\[
\begin{align*}
    f(y) &= \sqrt{y}, \\
    dY_t &= \alpha Y_t dt + \beta Y_t dW_t^y \quad \text{Lognormal,} \\
    d < W^y, W^x >_t &= 0.
\end{align*}
\]

Symmetric smiles; computation via conditioning on $\int_0^T f^2(Y_t)dt / T$. 
Assume $z = Z^a_0$ the initial point:

$$\frac{1}{T} \int_0^T F^2(Z^a_t) dt \xrightarrow{a \to 0} F^2(z).$$

Residual fluctuations:

$$\int_0^T \left( F^2(Z^a_t) - F^2(z) \right) dt$$

$$= \int_0^T \left[ F^2 \left( z + a \int_0^t ds + \sqrt{a} \int_0^t dW_s \right) - F^2(z) \right] ds$$

$$\sim 2F'(z)F(z) \left( \sqrt{a} \int_0^T W_t dt \right).$$

Remark again $\mathbb{E} \left[ \int_0^T W_t dt \mid \mathcal{F}_0 \right] = 0$.

Correction: Correlation with the BM driving the underlying gives correction at the order $\sqrt{a}$. 
Case (iiiB): Small Volatility Fluctuations

- Assume $z = Z_0^a$ the initial point:

$$\frac{1}{T} \int_0^T F^2(\delta Z_t^1) dt \xrightarrow{\delta \to 0} F^2(0).$$

Residual fluctuations:

$$\int_0^T \left( F^2(\delta Z_t^1) - F^2(0) \right) dt \sim 2F'(0)F(0) \left( \delta \int_0^T Z_t dt \right).$$

**Correction**: Correlation with the BM driving the underlying gives correction at the order $\delta$. 
Correction via Martingale Method

- The price: $P_t = \mathbb{E}[h(X_T) \mid \mathcal{F}_t]$, a semi-martingale, here with $r = 0$ a martingale.

- Introduce the approximation:

  $$\tilde{P}_t = M_t + R_t$$

  with Ansatz:
  (i) $\tilde{P}_t = \tilde{P}(t, X_t)$, $\tilde{P}_T = h(x)$, (ii) $M_t$ martingale, (iii) $R_t = O(\text{"small"})$.

  \[\rightarrow \text{Then:}\]

  $$\tilde{P}_t = M_t + R_t = \mathbb{E}[M_T \mid \mathcal{F}_t] + R_t$$

  $$= \mathbb{E}[M_T + R_T \mid \mathcal{F}_t] + R_t - \mathbb{E}[R_T \mid \mathcal{F}_t]$$

  $$= P_t + (R_t - \mathbb{E}[R_T \mid \mathcal{F}_t]) = P_t + O(\text{"small"}) .$$
Fast Scale Approximation in Mixing Case

\[ \tilde{P}_t - \tilde{P}_0 \overset{\text{Ito}}{=} \int_0^t \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s ds + F(Z_s^a)X_s \partial_x \tilde{P}_s dW_s^* \]

by Poisson

\[ = \int_0^t \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s ds + F(Z_s^a)X_s \partial_x \tilde{P}_s dW_s^{(x)} \]

by parts

\[ + \frac{1}{2a} \left( d\phi - \sqrt{a} \phi'(Z_s^a) dW_s \right) X_s^2 \partial_x^2 \tilde{P}_s \]

\[ + \ldots dW_s^* + \ldots dW_s + O(1/a) \]

objective = martingale + O(1/a).

\[ d\phi X_s^2 \partial_x^2 \tilde{P}_s = d \left( \phi X_s^2 \partial_x^2 \tilde{P}_s \right) - \phi d \left( X_s^2 \partial_x^2 \tilde{P}_s \right) - d \left\langle \phi, x^2 \partial_x^2 \tilde{P} \right\rangle_s, \]

\[ d \left\langle \phi, x^2 \partial_x^2 \tilde{P} \right\rangle_s = \sqrt{a} \phi'(Z_s^a) F(Z_s^a)(X_s \partial_x)(X_s^2 \partial_x^2) \tilde{P}_s ds \]

\[ = \sqrt{a} V(Z_s^a)(X_s \partial_x)(X_s^2 \partial_x^2) \tilde{P}_s ds. \]

\[ \leftrightarrow \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_t - 1/\sqrt{a} \left\langle V(Z_t) \right\rangle_{\Phi_Z} (x \partial_x)(x^2 \partial_x^2) \tilde{P}_t = 0; \tilde{P}_T = h. \]
The Fast Long Range Case

Recall key step in mixing argument, the Poisson step:

\[ \mathcal{L}_{Z^1} \phi = F^2(\bar{z}) - \bar{\sigma}^2, \quad \iff \left( F^2(Z_t^a) - \bar{\sigma}^2 \right) = \frac{1}{a} \left( d\phi(Z_t^a) - \sqrt{a} \phi'(Z_t^a) dW_t^{(1/2)} \right). \]

**Long range case is much harder!!** Useful quantity: the conditionally expected volatility fluctuations over remaining time epoch:

\[ \phi_t^a = \mathbb{E} \left[ \frac{1}{2} \int_t^T \left( (\sigma_s^a)^2 - \bar{\sigma}^2 \right) ds \bigg| \mathcal{F}_t \right] = O(“small”') \]

Decompose as

\[ \phi_t^a = \psi_t^a - \frac{1}{2} \int_0^t \left( F(Z_s^a) \right)^2 - \bar{\sigma}^2 \right) ds, \]

where the martingale \( \psi_t^a \) is defined by

\[ \psi_t^a = \mathbb{E} \left[ \frac{1}{2} \int_0^T \left( F(Z_s^a) \right)^2 - \bar{\sigma}^2 \right] ds \bigg| \mathcal{F}_t \right], \]

so that we have

\[ \left( F^2(Z_t^a) - \bar{\sigma}^2 \right) = 2 \left( d\psi(Z_t^a) - d\phi(Z_t^a) \right). \]
To identify problem for the correction we also need, as above, covariation with asset. We have:

$$d \langle \psi^a, W \rangle_t = \left(a^H - 1 \theta_t + \tilde{\theta}_t^a\right) dt,$$

for

$$\theta_t = (T - t)^{H-1/2} \frac{\sigma_{ou} \langle FF' \rangle}{\Gamma(H + 3/2)},$$

and $\tilde{\theta}_t^a$ is random but small:

$$\limsup_{a \to \infty} a^{1-H} \sup_{t \in [0,T]} \mathbb{E} \left[ (\tilde{\theta}_t^a)^2 \right]^{1/2} = 0.$$

Rapid volatility fluctuations reflects themselves in a small covariation.
To deal with a general (smooth) function of the volatility fluctuations and get bounds for these we use a Hermite decomposition for the stochastic volatility. We denote

$$\tilde{F}(z) = F(\sigma_{\text{ou}} z)^2.$$  

Because \( \mathbb{E}[\tilde{F}(Z)^2] < \infty \) is finite when \( Z \) is a standard normal variable, the function \( \tilde{F} \) can be expanded in terms of the Hermite polynomials

$$H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2}$$

and the series

$$\sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(z),$$

$$C_k = \mathbb{E}[H_k(Z)\tilde{F}(Z)] = \int_R H_k(z)\tilde{F}(z)p(z)dz,$$

converges in \( L^2(R, p(z)dz) \) to \( \tilde{F}(z) \).
Price Approximation Fast Long Range Case:

\[ P_t = Q_t^a(X_t) + o(a^{H-1}), \]

where

\[ Q_t^a(x) = Q_t^{(0)}(x) + (x^2 \partial_x^2 Q_t^{(0)}(x)) \phi_t^a + a^{H-1} \rho Q_t^{(1)}(x), \]

with

\[ \mathcal{L}_{BS}(\bar{\sigma}) Q_t^{(0)}(x) = 0, \quad Q_T^{(0)}(x) = h(x), \]

and \( \phi_t^\varepsilon \) is the random path dependent component

\[ \phi_t^\varepsilon = \mathbb{E} \left[ \frac{1}{2} \int_t^T \left( (\sigma_s^\varepsilon)^2 - \bar{\sigma}^2 \right) ds \mid \mathcal{F}_t \right], \]

moreover, \( Q_t^{(1)}(x) \) is the deterministic correction

\[ Q_t^{(1)}(x) = x \partial_x \left( x^2 \partial_x Q_t^{(0)}(x) \right) \left[ (T - t)^{H+\frac{1}{2}} \frac{\sigma_{ou} \langle F \rangle \langle FF' \rangle}{\Gamma(H + \frac{3}{2})} \right]. \]

- Case \( H = 1/2 \) gives deterministic \( \phi_t^\varepsilon \) and reordering of terms, but we have “continuity” with respect to qualitative form.
\[
\frac{1}{K} \left( \phi_t^\varepsilon \left( x^2 \partial_x^2 \right) Q_t^{(0)}(x) + a^{H-1} \rho Q_t^{(1)}(x) \right) \\
= \left( \frac{e^{-d_1^2 / 2 x^2 / K}}{\sqrt{\pi}} \right) \left\{ \frac{\phi_t^\varepsilon}{2} \left( \frac{\tau}{\bar{\tau}} \right)^{-1/2} + a_F \left[ \left( \frac{\tau}{\bar{\tau}} \right)^H + \left( \frac{\tau}{\bar{\tau}} \right)^{H-1} \log \frac{K}{x} \right] \right\},
\]

with \( \tau = T - t \) and the effective diffusion time \( \bar{\tau} = 2/\bar{\sigma}^2 \), moreover with:

\[
d_1 = \frac{\frac{\tau}{\bar{\tau}} - \log \frac{K}{x}}{\sqrt{2 \frac{\tau}{\bar{\tau}}}} \]

\[
a_F = a^{H-1} \rho \sigma_{ou} \left\{ \langle F \rangle \langle FF' \rangle \bar{\tau}^H \right\} \\
2^{3/2} \bar{\sigma} \Gamma(H + \frac{3}{2})
\]

\[
\phi_t^\varepsilon = O \left( \left( \frac{\varepsilon}{\bar{\tau}} \right)^{1-H} \left( \frac{\tau}{\bar{\tau}} \right)^H \right).
\]
The figure shows the relative price correction as function of relative time to maturity $\tau/\bar{\tau}$ for 3 values of the moneyness $K/x$. The solid lines is the mean price and the dashed lines gives the mean price plus/minus one standard deviation. We used here $a_F = 0.1$ and $\left( \frac{1}{a\bar{\tau}} \left( 1 - H \right) \bar{\tau} \sigma_\phi \right)^{1/2} = 0.02$. 
Implied Volatility Fast Long Range Call Case

\[ I_t = \bar{\sigma} + \frac{\phi_{\varepsilon}}{\bar{\sigma}(T-t)} + \bar{\sigma}a_F \left[ \left( \frac{\tau}{\bar{\tau}} \right)^{H-1/2} + \left( \frac{\tau}{\bar{\tau}} \right)^{H-3/2} \log \left( \frac{K}{X_t} \right) \right] + o(\varepsilon^{1-H}) \]

\[ = \mathbb{E} \left[ \frac{1}{T-t} \int_t^T (\sigma_s^\varepsilon)^2 ds \right]^{1/2} \left[ \left( \frac{\tau}{\bar{\tau}} \right)^{H-1/2} + \left( \frac{\tau}{\bar{\tau}} \right)^{H-3/2} \log \left( \frac{K}{X_t} \right) \right] + o(\varepsilon^{1-H}). \]

→ We get fractional correction to expected volatility fluctuations, moreover, the correction diverges in the long maturity limit when the volatility fluctuations have correlations that decays slowly (as a fractional power).
The figure shows the implied volatility correction as function of relative time to maturity $\tau/\bar{\tau}$ for 3 values of the moneyness $K/x$. The solid lines is the mean correction and the dashed lines gives the mean correction plus/minus one standard deviation, parameters as above.
As $a \to \infty$, the random process $a^{1-H}\phi^a_{t,T}$, $t \leq T$, converges in distribution (in the sense of finite-dimensional distributions) to a centered Gaussian random process $\phi_{t,T}$, $t \leq T$, we have the general covariance covariance function of $\phi^a_{t,T}$ has the following limit for any $t \leq T$, $t' \leq T'$, with $t \leq t'$: 

$$d^{2-2H}\mathbb{E}[\phi^a_{t,T}\phi^a_{t',T'}] \xrightarrow{a \to \infty} \sigma^2_\phi(T-t)^H(T'-t')^H C_\phi(t,t';T,T'),$$

with $\sigma^2_\phi = \sigma^2_{ou} \langle FF'\rangle^2 \left(1/\Gamma(2H+1) \sin(\pi H) - 1/2H \Gamma(H + \frac{1}{2})^2 \right)$, and where the limit correlation is

$$C_\phi(t,t';T,T') = \frac{\int_0^\infty du \left[(u+r)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right] \left[(u+s)^{H-\frac{1}{2}} - (u+q)^{H-\frac{1}{2}}\right]}{\int_0^\infty du \left[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]^2},$$

with

$$q = \frac{t'-t}{\sqrt{(T-t)(T'-t')}}, \quad r = \frac{\sqrt{T-t}}{\sqrt{T'-t'}}, \quad s = \frac{T'-t}{\sqrt{(T-t)(T'-t')}},$$
The figure shows how the stochastic correction coefficient decorrelate. For $T = T'$ we show $C_\phi(t, t', T, T')$ as function of $|\tau - \tau'|/|\tau + \tau'|$. 
The figure shows how the stochastic correction coefficient decorrelate with respect to time translation. We consider the case with $T - t = T' - t'$ and plot the correlation coefficient as function of $|t' - t| / |T - t|$. Note the rapid decay at the origin followed by a long range behavior. This shows how the price correction surface decorrelate as we move in time.
Implied Volatility Small Fractional Volatility Fluctuations:

\[ I_t = \mathbb{E} \left[ \frac{1}{T-t} \int_t^T F(Z_s^q)^2 \, ds \mid \mathcal{F}_t \right]^{\frac{1}{2}} + \mathcal{A}(T-t) \left[ 1 + \frac{\log(K/X_t)}{(T-t)/\bar{\tau}} \right], \]

for

\[ \mathcal{A}(\tau) = \frac{\delta \rho \bar{\sigma} \tau^{H+\frac{1}{2}}}{2 \Gamma(H+\frac{5}{2})} \left\{ 1 - \int_0^{a\tau} e^{-v} \left( 1 - \frac{v}{a\tau} \right)^{H+\frac{3}{2}} \, dv \right\}, \]

In the short and long time to maturity regimes we then have for the leverage term:

\[ \mathcal{A}(\tau) \left[ 1 + \frac{\log(K/X_t)}{\tau/\bar{\tau}} \right] = \begin{cases} a_s \left[ (\tau/\bar{\tau})^{\frac{1}{2}+H} + (\tau/\bar{\tau})^{-\frac{1}{2}+H} \log(K/X_t) \right] & \text{for } a\tau \ll 1, \\ a_1 \left[ (\tau/\bar{\tau})^{-\frac{1}{2}+H} + (\tau/\bar{\tau})^{-\frac{3}{2}+H} \log(K/X_t) \right] & \text{for } a\tau \gg 1, \end{cases} \]
Remark on Accuracy Call Case

Note that in the call case there is a singular payoff function.

One has to show that $\mathbb{E}[R_T - R_t | \mathcal{F}_t]$ is “small” which involves bounds on higher order derivatives of the Black-Scholes price and showing that the local martingale introduced in the price approximation derivation indeed are martingales (up to time $T$).
• Have analyzed **fractional stochastic volatility**.

• Analysis very different from (pde approach in) mixing case, main tool used is Martingale Method.

• Framework general.

• Long range correlation gives a random implied volatility surface with fractional term structure.

• Calibration parameters provide linkage between financial products.