Math 126, Spring 2024
Final Exam

Name: (please print) ______________________

Signature: ______________________

Student ID Number: ______________________

Circle your Class below:

Li 9am	Haskell 9am	Haskell 10am	Garzoni 11am
Tabing 12pm	Li 1pm	Other	

Instructions:

- Before you start the test, initial the front, top-right corner of every page. (These tests will have the staple cut off in order to scan them, so, in case of mishaps, it is imperative you identify yourself on every page.)
- You have 2 hours to complete this exam. You may use a handwritten sheet of notes (8.5×11, front and back). The notes must be in your own handwriting and written with a pen or pencil. Notes created on a computer and printed out are not permitted. Notes that are photocopied are not permitted. You may not use a magnifying glass. You may not use a calculator.
- If you need extra space for any problem, you may write on the back of the test, which is blank. If you do this, you *must* indicate that you have done so; any work written there may not be seen by the grader unless you write on the page where the problem is stated that the work continues there. You must also indicate there, to which problem(s) the work pertains.
- Unless explicitly stated otherwise, you must clearly show each step of your solution for full credit. Correct answers without any work will not receive credit. Some questions may remind you to show your work explicitly, but you should assume each question requires you to justify your steps by making your thought process clear in your solution.
- All work you submit should represent your own thoughts and ideas. If the graders suspect otherwise, you can expect your instructor to file a report with the USC Office of Academic Integrity.
- This exam has 10 problems with a total of 140 points.

Problem 1. *(12 points)*
Evaluate the following limits.

(a) $\lim_{x \to \infty} x \sinh\left(\frac{1}{x}\right)$

(b) $\lim_{x \to 0^+} x^{\sin x}$

Problem 2. *(24 points)*
Find the following antiderivatives.

(a) $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

This problem is continued on the next page.

Problem 2 cont.

(b) $\displaystyle\int \frac{x+1}{x^3+x}\,dx$

Problem 3. *(10 points)*
Consider the improper integral

$$\int_0^1 \frac{5-\sin x}{x^{1/3}+x^3}\,dx.$$

A student, investigating this integral, makes the following three claims.

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1) $\dfrac{5-\sin x}{x^{1/3}+x^3} \leq \dfrac{6}{x^3}$ when $0 < x \leq 1$.
2) $\displaystyle\int_0^1 \frac{6}{x^3}\,dx$ is convergent.
3) It follows from Claims 1 and 2 that the integral is convergent.

(a) Identify the flaw in the student's argument. Circle the best response.

- (i) Claim 1 is false.
- (ii) Claim 1 is true but claim 2 is false.
- (iii) Claims 1 and 2 are true but it does not follow from them that the integral is convergent.

(b) Is the integral convergent or divergent? Carefully justify your answer.

Problem 4. *(16 points)*
Consider the region in the first quadrant that is bounded by $y = \cos x$, $x = 0$, and $y = 0$ and the solid that is formed by rotating this region about the y-axis.

(a) Find the volume of the solid.

(b) Find a definite integral whose value is equal to the surface area of the curved surface of the solid (not including the circular flat base). You do not need to evaluate the integral.

Problem 5. *(12 points)*
A container full of fuel is in the shape of the region in the first quadrant bounded by $y = (x+1)^{1/2}$, $x = 0$ and $y = 4$ rotated about the y-axis. Find a definite integral whose value is the work needed to pump the fuel out of the tank. Assume the values of x and y are in meters, the density of the fuel is ρ kg/m^3, and the acceleration due to gravity is g m/s^2. You do not need to evaluate the integral.

Problem 6. *(6 points)*

Find the sum of the series $\sum_{n=0}^{\infty} \frac{3(-1)^n}{2^{2n+1}}$.

Problem 7. *(18 points)*
Determine the convergence of each series. Circle your answer in each case and justify it carefully, clearly stating any tests you use.

(a) $\displaystyle\sum_{n=0}^{\infty}(\sin n)\left(\frac{n+3}{5n+7}\right)^{2n}$

Absolutely Convergent | Conditionally Convergent | Divergent

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$

Convergent | Divergent

Problem 8. *(20 points)*
Consider the function m defined by the power series

$$m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2n+1} = 1 - \frac{x^4}{3} + \frac{x^8}{5} - \frac{x^{12}}{7} + \frac{x^{16}}{9} - \dots$$

(a) Find the radius of convergence of m.

(b) Find the interval of convergence of m.

(c) Find $m^{(24)}(0)$.

This problem is continued on the next page.

Problem 8 cont.

$$m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2n+1} = 1 - \frac{x^4}{3} + \frac{x^8}{5} - \frac{x^{12}}{7} + \frac{x^{16}}{9} - \dots$$

(d) Find the value of

$$\int_0^{1/2} m(x)\, dx$$

with an error of less than $0.001 = 1/1000$. Leave your answer as a finite sum of fractions.

(e) What is the function $m(x)$? Find an expression for $m(x)$ using familiar functions.

Problem 9. *(12 points)*
Let $T_3(x)$ denote the Taylor polynomial of degree at most 3 about $x = \pi/4$ of $f(x) = \tan x$. Notice the first few derivatives of f are:

$$\begin{aligned} f(x) &= \tan x \\ f'(x) &= \sec^2 x = 1 + \tan^2 x \\ f''(x) &= 2\tan x + 2\tan^3 x \\ f'''(x) &= 6\tan^4 x + 8\tan^2 x + 2 \\ f^{(iv)}(x) &= 24\tan^5 x + 40\tan^3 x + 16\tan x \end{aligned}$$

You do not need to show these.

(a) Find $T_3(x)$.

(b) Find a bound on the error when $T_3(x)$ is used to approximate $\tan(x)$ and $0 \leq x \leq \pi/4$.

Problem 10. *(10 points)*

(a) A power series $\sum_{n=0}^{\infty} a_n(x-7)^n$ converges when $x = 3$ and diverges when $x = 12$. Does the series converge when $x = 8$? Circle the best option and explain briefly.

(i) It converges absolutely.

(ii) It converges though it may only converge conditionally.

(iii) It diverges.

(iv) There is not enough information to determine this.

(b) Consider the power series $\sum_{n=0}^{\infty} a_n(x-9)^n$. Suppose

$$\lim_{n\to\infty} \frac{|a_n|}{7^n} = 4.$$

What is the radius of convergence of the power series, R? Choose one option and explain briefly.

(i) 1/7

(ii) 1/4

(iii) 4

(iv) 7

(v) 9

(vi) There is not enough information to determine this.