Math 126 Final Examination, Spring 2002

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1. (30 points) Evaluate the integrals.

   a) $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx$

   b) $\displaystyle\int \frac{4}{x^3+x}\,dx$

2. (10 points) Determine whether the integral is convergent or divergent. Clearly explain your reasoning.
$$\int_{\pi}^{\infty} \frac{1+\sin x}{x^{3/2}}\,dx$$

3. (15 points) Consider the curve described by the polar equation $r = 1 + 2\cos\theta$ that is sketched below. Find the area of the inner loop.

4. (15 points) Find the volume of the solid that is obtained when the region bounded by the curves $y = x^2$ and $y = 4x$ is rotated about the y-axis.

5. (10 points) Determine if the sequence $\left\{a_n = \left(1+\frac{2}{n}\right)^{-n}\right\}$ converges or diverges. If it converges, find its limit.

6. (10 points) Determine if the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n^2+5}{3n^2-2}$ is absolutely convergent, conditionally convergent, or divergent. Clearly state any test you use and show that all the necessary conditions for applying that test are satisfied.

7. (20 points) Determine if each series is convergent or divergent. Clearly state any test you use and show that all the necessary conditions for applying that test are satisfied.

   a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^2+3n+2}$

   b) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$

8. (20 points) Consider the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(x+2)^n}{\sqrt{n}3^n}$.

   a) Find its radius of convergence.

   b) Find its interval of convergence (i.e. determine all values of x for which the series converges.)

9. (20 points)

   a) Write down the Maclaurin series of $f(x) = e^{-x^2}$ and of $g(x) = \frac{1}{1+x^2}$. Include at least the first four non-zero terms.

b) Using your answer to (a) determine which function is larger when x is close (but not equal) to 0.

10. (20 points)

a) Write down the Taylor series about 0 (i.e. the Maclaurin series) for the function $f(x) = x^2 e^{-x}$. Include at least the first four non-zero terms.

b) If we approximate $f(1/10)$ using the 3rd degree Taylor polynomial, $T_3(1/10)$, find an upper bound for the error in the approximation. Explain.

11. Choose TWO of the following problems and do one of them on each of the following blank pages.

A. A tank has the shape of the solid that is obtained by rotating $y = x^2$, $0 < x < 1$ about the y-axis. The tank is full of water. Write, but do not evaluate, an integral that is the work required to pump all of the water out of the tank. The units of x and y are meters.

B. The density of an oil slick on the surface of the ocean at a distance r meters from the site of the spill is $\rho(r) = \frac{50}{1+r}$ kg/m^2. If the slick extends from $r = 0$ to $r = 10,000$ find the total mass of oil in the slick.

C. A pool of water is 15 feet deep. A cylindrical drum of radius 2 feet is lying on its side at the bottom of the pool. Write, but do not evaluate, an integral that is the hydrostatic force on one end of the drum.

D. Suppose that the sizes of files available on the Internet are distributed according to the probability density function

$$f(x) = \begin{cases} \frac{1.5(10)^{1.5}}{x^{2.5}} & x \geq 10 \\ 0 & x < 10, \end{cases}$$

where x denotes the file size in Kbytes. Compute the probability of a file with size larger than 1000 Kbytes.

E. Find the solution of the differential equation that satisfies

$$\frac{dy}{dx} = \frac{1+x}{xy}, \quad x > 0, \quad y(1) = -4$$