MATH 126 **FINAL EXAM** **December 15, 2010**

INSTRUCTIONS

Read the problems carefully and answer the questions asked. Write neatly and indicate clearly your answer to each problem. The backs of the sheets may be used for scratch paper or to continue your work on a problem. You are to do all ten problems. You must show your work to obtain full credit. Points may be deducted if you do not justify your final answer. You are allowed to use your own hand-written formula sheet. Any other notes, books, calculators or collaboration with others are *not* allowed. If you have any questions about any of the problems ask the proctor, but no one else!

1. [20 points] Determine whether or not the limit exists. If the limit exists find it, and indicate clearly how you obtained your answer. If the limit does not exist give reasons why.

(i) $\displaystyle \lim_{x\to 0} \frac{\sin 5x}{\tan 3x}$; (ii) $\displaystyle \lim_{x\to 0} (1-3x)^{1/x}$; (iii) $\displaystyle \lim_{n\to\infty} \frac{n+(-1)^n\sqrt{n}}{n+\sqrt{n}}$.

2. [30 points] Evaluate the integrals.

(i) $\displaystyle \int e^{2x}\cos\left(e^x\right)\,dx$; (ii) $\displaystyle \int \frac{x^3}{\sqrt{1-x^2}}\,dx$; (iii) $\displaystyle \int \frac{x-1}{(x+2)(x^2+1)}\,dx$.

3. [20 points] Consider the region $\mathcal{R}$ bounded by the line $y=3x$ and the parabola $y=2+x^2$.

(i) Sketch the region $\mathcal{R}$.

(ii) Find the volume of the solid obtained by rotating the region $\mathcal{R}$ about the y-axis.

(iii) Set up but DO NOT EVALUATE an integral giving the volume of the solid obtained by rotating the region $\mathcal{R}$ about the line $y=1$.

4. [15 points] A tank has the shape obtained by rotating about the y-axis the region bounded by $y=x^2$, $x=0$, $y=1$, and $y=9$, where x and y are measured in meters. Assume that the tank is filled with liquid up to the level $y=7$.

(i) Draw a sketch of the tank.

(ii) Find an integral that expresses the work done in pumping all of the liquid out of the top of the tank. DO NOT EVALUATE the integral. Use ρ to denote the density of the liquid (in kg/m^3) and g to denote the acceleration due to gravity (in m/s^2).

5. [20 points] The time rate of change of a population of bacteria grown in a controlled laboratory environment is modelled by the differential equation

$$\frac{dy}{dt} = \frac{1}{2}t(170 - y),$$

where y is the population of the bacteria and t is the elapsed time in hours. The initial population of bacteria is $y = 10$.

(i) Find the population at time t.

(ii) Find the population in an equilibrium state, that is, find the value $\lim_{t\to\infty} y(t)$.

6. [20 points] Determine whether the series is convergent or divergent. Remember to justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(100n)}{(1.01)^n}$; (ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$; (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

7. [15 points] Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n3^n}.$$

8. [20 points] (i) Write down the Maclaurin series for the function

$$f(x) = \sqrt[3]{1+x}.$$

You should give the first three non-zero terms in the series explicitly, as well as an expression for the general term.

(ii) Use your answer to (i) to obtain a series expansion for

$$\int_0^{1/2} x\,\sqrt[3]{1+x^3}\,dx$$

[If you do not know the answer to (i) you should write $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and then give your answer to (ii) in terms of the a_n.]

9. [20 points] (i) Approximate the function $f(x) = x^{2/3}$ by its Taylor polynomial of degree 2 at $a = 8$.

(ii) Use Taylor's formula to determine the accuracy of this approximation at the point $x = 7.5$. (You do not need to simplify your answer.)

10. [20 points] Let $\mathcal{D}$ be the region inside the curve $r = 2\cos\theta$ and outside the curve $r = 1$. Sketch the region $\mathcal{D}$ and find its area.