MATH 125, FINAL EXAM (common) **December 2005**

1. (6 points each) Calculate the following limits.

a) $\lim_{x \to 4} \dfrac{x-4}{\sqrt{x}-2}$.

b) $\lim_{x \to 1} \dfrac{e^x - e}{x-1}$.

c) $\lim_{x \to 0} \dfrac{3x + x^2}{\sin x}$.

2. (7 points each) Find $\frac{dy}{dx}$.

a) $y = \ln(2x^2 - 3x)$.

b) $y = \dfrac{2x-3}{e^{x+1}}$.

c) $y = x^{\sin x}$ (Use logarithmic differentiation).

d) $y = \int_0^{\tan x} \sqrt{t^2+4}\,dt$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

3. (8 points each) Evaluate the following integrals:

(a) $\int_1^2 x^2(x-2)^{\frac{2}{5}}\,dx$.

(b) $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}\,dx$.

(c) Find the area of the region bounded by the curves $y = 0$, $y = xe^{x^2}$, $x = 0$ and $x = 1$.

4. Consider the following function and its first and second derivative:

$$f(x) = \frac{x}{x^2+4} \qquad f'(x) = -\frac{x^2-4}{(x^2+4)^2} \qquad f''(x) = \frac{2x(x^2-12)}{(x^2+4)^3}.$$

a) (5 points) Find the critical numbers of f.

b) (10 points) Determine where f is increasing, where f is decreasing, and find the local maxima and minima of f.

c) (5 points) Find the asymptotes of f.

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d) (5 points) Find the inflection points of f and determine where f is concave upwards and downwards.

e) (8 points) Draw a rough sketch of the graph of f.

5. (12 points) a) Find the absolute maximum and minimum of $f(x) = xe^{-3x}$ on the interval $[-1, 1]$.
b) (8 points) For what values of c is the function $f(x)$ continuous everywhere?

$$f(x) = \begin{cases} cx + 1 & x \leq 3 \\ cx^2 - 1 & x > 3 \end{cases}$$

6. a) (10 points) Show that the equation $\cos x = 2x$ has a solution.
b) (10 points) Show that it has exactly one solution.

7. (20 points) A cylindrical can is to be constructed so that its volume will be $40cm^3$. If the material used to make the top and bottom is twice as expensive as the material used to make the side, find the dimensions of the least expensive can.

8. (17 points) A tank in the shape of an inverted circular cone is standing on its tip. It has a diameter of $4m$, and a height of $9m$. If water leaks out at a rate of $2m^3/min$ how fast is the water level falling when the water level is $3m$? (If r is the radius and h the height, then the volume of a cone is $\frac{1}{3}\pi r^2 h$).

9. (20 points) Use the tangent line approximation (linear approximation) at $x = 27$ to estimate the cube root of 26.