This exam has 9 problems, and will last 120 minutes.

You may not use any type of calculator. No extra notes are allowed.

Show all of your work and justify every answer to receive full credit.

Feel free to continue answers on other pages as long as you clearly indicate to the grader where they can find your solution.

Work quickly, but carefully. Good luck!
Trig Identities

\[
\begin{align*}
\sin(2\alpha) &= 2 \sin \alpha \cos \alpha \\
\cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\
\cos(2\alpha) &= 1 - 2\sin^2 \alpha \\
\cos(2\alpha) &= 2\cos^2 \alpha - 1 \\
\tan(2\alpha) &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \\
\sin^2 \alpha &= \frac{1 - \cos 2\alpha}{2} \\
\cos^2 \alpha &= \frac{1 + \cos 2\alpha}{2} \\
\end{align*}
\]

Integral formulas

\[
\begin{align*}
\int \tan x \, dx &= \ln |\sec x| \\
\int \sec x \, dx &= \ln |\sec x + \tan x| \\
\int u \, dv &= uv - \int v \, du \\
\int \frac{a}{a^2 + x^2} \, dx &= \tan^{-1} \frac{x}{a} \\
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx &= \sin^{-1} \frac{x}{a} \\
\int \ln x \, dx &= x \ln x - x
\end{align*}
\]

Cylindrical coordinates: \( x = r \cos \theta, \ y = r \sin \theta, \ z = z \)

If \( \mathbf{r}(u, v) = A \cos u \, \mathbf{i} + A \sin u \, \mathbf{j} + v \, \mathbf{k} \), then

\[
\begin{align*}
\mathbf{r}_u \times \mathbf{r}_v &= A \cos u \, \mathbf{i} + A \sin u \, \mathbf{j} \\
\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| &= A
\end{align*}
\]

Spherical coordinates: \( x = \rho \sin \phi \cos \theta, \ y = \rho \sin \phi \sin \theta, \ z = \rho \cos \phi \).

If \( \mathbf{r}(\phi, \theta) = A \sin \phi \cos \theta \, \mathbf{i} + A \sin \phi \sin \theta \, \mathbf{j} + A \cos \phi \, \mathbf{k}, \) then

\[
\begin{align*}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= A^2 \sin^2 \phi \cos \theta \, \mathbf{i} + A^2 \sin^2 \phi \sin \theta \, \mathbf{j} + A^2 \sin \phi \cos \phi \, \mathbf{k} \\
\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| &= A^2 \sin \phi
\end{align*}
\]
Question 1. (10pts) Let \( a = \langle -1, 2, 3 \rangle, b = \langle 2, -2, 0 \rangle, c = \langle 3, 1, 4 \rangle \), and let \( A, B, C \) be the corresponding points.

(a) Find the scalar and vector projections of \( a \) in the direction of \( c \) (\( \text{comp}_c a \) and \( \text{proj}_c a \)).

(b) Does the triangle in \( \mathbb{R}^3 \) with vertices at \( A, B, C \) contain an angle larger than \( 90^\circ \)? Justify your answer using vectors.
Question 2. (15pts) Let $f(x, y, z) = x^2 + e^{-yz}$.

(a) Find the directional derivative of $f$ at the point $(1, 0, 1)$ in the direction of the unit vector $\frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j}$. (Note: $\frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j} = \frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j} + 0 \mathbf{k}$)

(b) The equation $x^2 + e^{-yz} = 2$ represents a level surface $S$ of the function $f(x, y, z)$. Find the equation of the tangent plane to $S$ at the point $(1, 0, 1)$.

(c) Let $g(s, t) = f(\cos t, \sin^2 t, s^3)$ where $f(x, y, z) = x^2 + e^{-yz}$. Find $\frac{\partial g}{\partial s}(1, 0)$ and $\frac{\partial g}{\partial t}(1, 0)$. 
Question 3. (10pts) Use the Lagrange multipliers to find the maximum and minimum of the function $f(x, y, z) = xyz$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 6$. 
Question 4. (10pts) Find all the critical points of the function \( f(x,y) = e^x - x - \cos y \). Classify them as local minimum, local maximum or saddle point. You must justify your answer to get full points. Note that there is more than one critical point.
Question 5. (15pts) Answer the following.

(a) Evaluate the double integral
\[
\int_0^1 \int_{\sqrt[3]{x}}^1 e^{y^4} dy \, dx.
\]

(b) Rewrite the triple integral
\[
\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^3 \cos \theta \sin^2 \phi \, d\rho \, d\phi \, d\theta
\]

in Cartesian coordinates in the order \(dz \, dy \, dx\). Do not evaluate the integral.
Question 6. (15pts) Answer the following.

(a) Evaluate
\[ \int_C (x + 2y)dx + (x^2 - y^2)dy \]  \hspace{1cm} (1)
where \( C \) is the line segment \( C_1 \) from \((0,0)\) to \((1,0)\) followed by the line segment \( C_2 \) from \((1,0)\) to \((1,1)\).

(b) Evaluate the integral (1) over the arc \( C_3 \) of the parabola \( y = x^2 \) from \((0,0)\) to \((1,1)\).

(c) Are the results in parts (a) and (b) the same? Explain why.
Question 7. (10pts) Let $C$ be the curve depicted in the figure below with the indicated orientation. Compute the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y) = (e^x - 2xy, 2x - x^2)$. 

![Diagram of the curve C with points (-1,0), (1,0), (-1,2), (1,2), (0,3) labeled.](Image)
Question 8. (20pts) Suppose that $S$ is the upper cap of the sphere $x^2 + y^2 + z^2 = 169$ that lies above the plane $z = 12$, and suppose that $S$ has an upward (positive $z$) orientation.

Also consider the vector field $\mathbf{F}(x, y, z) = \langle 2z, -4x, 3y \rangle$.

(a) State Stokes’ Theorem, and explain how it applies to this surface and vector field.

(b) Compute (directly) the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the boundary of $S$.

(c) Verify Stokes’ Theorem by also computing the corresponding surface integral.
You may continue your work for Question 8
Question 9. (15pts) Use the divergence theorem to evaluate

\[ \iiint_S \mathbf{F} \cdot d\mathbf{S} \]

where \( \mathbf{F} = xye^{y^2} \mathbf{i} - \frac{1}{2} e^{y^2} \mathbf{j} + z \mathbf{k} \)

where the surface \( S \) consists of the three parts: \( z = 4 - 3x^2 - 3y^2, 1 \leq z \leq 4 \) on the top, \( x^2 + y^2 = 1, 0 \leq z \leq 1 \) on the sides, and \( z = 0 \) on the bottom.