Math 226: Calculus III — Spring 2019

Final Exam — 08 May 2019

First Name: (as in student record)

Last Name: (as in student record)

USC ID: Signature:

Please circle your lecture time:

9am Mancera	10am Mancera	11am Jang	12pm Jang	12pm Rooney	1pm Tokorcheck

- This exam has 9 problems, and will last 120 minutes.
- You may not use any type of calculator. No extra notes are allowed.
- Show all of your work and justify every answer to receive full credit.
- Feel free to continue answers on other pages as long as you clearly indicate to the grader where they can find your solution.
- Work quickly, but carefully. Good luck!

Do not write in the box below:

Q01	Q02	Q03	Q04	Q05	Partial
/10	/15	/10	/10	/15	/60
Q06	Q07	Q08	Q09		Partial
/15	/10	/20	/15		/60

/120

Trig Identities

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\cos(2\alpha) = 1 - 2\sin^2\alpha$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1$$

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Integral formulas

$$\int \tan x\,dx = \ln|\sec x|$$

$$\int \sec x\,dx = \ln|\sec x + \tan x|$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$$

$$\int \ln x\,dx = x\ln x - x$$

Cylindrical coordinates: $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

If $\mathbf{r}(u,v) = A\cos u\,\mathbf{i} + A\sin u\,\mathbf{j} + v\,\mathbf{k}$, then

$$\mathbf{r}_u \times \mathbf{r}_v = A\cos u\,\mathbf{i} + A\sin u\,\mathbf{j}$$

$$\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = A$$

Spherical coordinates: $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$

If $\mathbf{r}(\phi,\theta) = A\sin\phi\cos\theta\,\mathbf{i} + A\sin\phi\sin\theta\,\mathbf{j} + A\cos\phi\,\mathbf{k}$, then

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = A^2\sin^2\phi\cos\theta\,\mathbf{i} + A^2\sin^2\phi\sin\theta\,\mathbf{j} + A^2\sin\phi\cos\phi\,\mathbf{k}$$

$$\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = A^2\sin\phi$$

Question 1. (10pts) Let $\mathbf{a} = \langle -1, 2, 3 \rangle$, $\mathbf{b} = \langle 2, -2, 0 \rangle$, $\mathbf{c} = \langle 3, 1, 4 \rangle$, and let A, B, C be the corresponding points.

(a) Find the scalar and vector projections of $\mathbf{a}$ in the direction of $\mathbf{c}$ ($\text{comp}_{\mathbf{c}}\mathbf{a}$ and $\text{proj}_{\mathbf{c}}\mathbf{a}$).

(b) Does the triangle in $\mathbb{R}^3$ with vertices at A, B, C contain an angle larger than $90°$? Justify your answer using vectors.

Question 2. (15pts) Let $f(x, y, z) = x^2 + e^{-yz}$.

(a) Find the directional derivative of f at the point $(1, 0, 1)$ in the direction of the unit vector $\frac{4}{5}\,\mathbf{i} + \frac{3}{5}\,\mathbf{j}$. (*Note:* $\frac{4}{5}\,\mathbf{i} + \frac{3}{5}\,\mathbf{j} = \frac{4}{5}\,\mathbf{i} + \frac{3}{5}\,\mathbf{j} + 0\,\mathbf{k}$)

(b) The equation $x^2 + e^{-yz} = 2$ represents a level surface S of the function $f(x, y, z)$. Find the equation of the tangent plane to S at the point $(1, 0, 1)$.

(c) Let $g(s, t) = f(\cos t, \sin^2 t, s^3)$ where $f(x, y, z) = x^2 + e^{-yz}$. Find $\frac{\partial g}{\partial s}(1, 0)$ and $\frac{\partial g}{\partial t}(1, 0)$.

Question 3. (10pts) Use the Lagrange multipliers to find the maximum and minimum of the function $f(x, y, z) = xyz$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 6$.

Question 4. (10pts) Find all the critical points of the function $f(x, y) = e^x - x - \cos y$. Classify them as local minimum, local maximum or saddle point. You must justify your answer to get full points. *Note that there is more than one critical point.*

Question 5. (15pts) Answer the following.

(a) Evaluate the double integral
$$\int_0^1 \int_{\sqrt[3]{x}}^1 e^{y^4}\, dy\, dx.$$

(b) Rewrite the triple integral
$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 \rho^3 \cos\theta \sin^2\phi \; d\rho\, d\phi\, d\theta$$
in Cartesian coordinates in the order $dz\, dy\, dx$. Do not evaluate the integral.

Question 6. (15pts) Answer the following.

(a) Evaluate

$$\int_C (x + 2y)dx + (x^2 - y^2)dy \tag{1}$$

where C is the line segment C_1 from $(0, 0)$ to $(1, 0)$ followed by the line segment C_2 from $(1, 0)$ to $(1, 1)$.

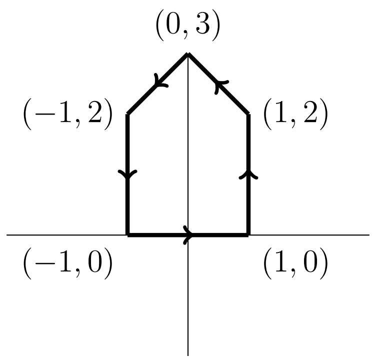
(b) Evaluate the integral (1) over the arc C_3 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

(c) Are the results in parts (a) and (b) the same? Explain why.

Question 7. (10pts) Let C be the curve depicted in the figure below with the indicated orientation. Compute the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y) = \langle e^x - 2xy, 2x - x^2 \rangle$.



Question 8. (20pts) Suppose that S is the upper cap of the sphere $x^2 + y^2 + z^2 = 169$ that lies above the plane $z = 12$, and suppose that S has an upward (positive z) orientation.

Also consider the vector field $\mathbf{F}(x, y, z) = \langle\, 2z,\ -4x,\ 3y \,\rangle$.

(a) State **Stokes' Theorem**, and explain how it applies to this surface and vector field.

(b) Compute (directly) the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of S.

(c) Verify Stokes' Theorem by also computing the corresponding surface integral.

You may continue your work for Question 8

Question 9. (15pts) Use the divergence theorem to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \quad \text{where} \quad \mathbf{F} = xye^{y^2}\,\mathbf{i} - \frac{1}{2}e^{y^2}\,\mathbf{j} + z\,\mathbf{k}$$

where the surface S consists of the three parts: $z = 4 - 3x^2 - 3y^2$, $1 \leq z \leq 4$ on the top, $x^2 + y^2 = 1$, $0 \leq z \leq 1$ on the sides, and $z = 0$ on the bottom.

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