Math 226 Final Exam
November 18th, 2020

Directions: Make sure that your student ID is properly inputted into the Math 226 Fall 2020 Final Exam Gradescope course. You must **show all of your work and justify your methods** to obtain full credit. In particular, if you use technology for any portion of a problem, you should indicate this so that we can understand your workflow and rule out foul play. Circle your final answers. Simplify your answers (unless the instructions indicate that it is unnecessary to do so). You are allowed to consult your notes or other non-human resources during the exam. Remember, USC considers cheating to be a serious offense; the minimum penalty is failure for the course. Cheating includes communication about the exam in any form with another human during the exam, or communication about the exam among students in staggered time zones during the time window that spans and includes exam times.

Questions: The **nine** exam questions will be presented twice, the first time in a consolidated format, and the second time, spaced apart for students who prefer printing to write the exam. Please make sure to tag each individual question when submitting through gradescope.

1. (10 points) Find an equation for the plane in $\mathbb{R}^3$ that contains the points $(0, 1, 2)$ and $(1, 2, 1)$ and that is parallel to the line $\vec{r}(t) = \langle 1 + 2t, \ -t, \ -4 - t \rangle$.

2. (10 points) Consider the two curves in $\mathbb{R}^3$ parametrized by

$$\vec{r_1}(t) = \langle t^2, \ t^2 + 2t, \ t^2 - 1 \rangle$$
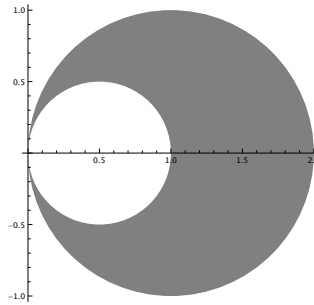
and

$$\vec{r_2}(u) = \langle u^2 + u + 1, \ u + 3, \ u^2 \rangle.$$

   (a) Find the (x, y, z) coordinates of the unique point where these two curves intersect.

   (b) At this point of intersection, find the angle between the two curves (the angle between their tangent lines). You may leave your answer in terms of the inverse cosine function.

3. (10 points) Suppose the temperature at a point in space is given by

$$T(x, y) = \frac{60}{1 + x^2 + 2y^2}.$$

   (a) (5 points) Find the direction of most rapid temperature increase at the point $(1, -1)$.

   (b) (5 points) Find the maximum rate of temperature increase at the point $(1, -1)$.

4. (10 points) Let $f(x, y) = x^4 + xy + y^4 + 1$.

   (a) (5 points) Find and classify the critical points of f as local maxima, local minima, or saddle points.

   (b) (5 points) Find the absolute maximum and the minimum values of $f(x, y)$ on the region defined by $x^4 + y^4 \leq 1$.

5. (10 points) Let D be the shaded region below (the area inside the circle of radius 1 centered at $(1,0)$, and outside the circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$).

(a) (5 points) Express $\iint_D f(x,y)\,dA$ as the sum of three double integrals in Cartesian coordinates, each in the order $dy\,dx$.

(b) (5 points) Express $\iint_D \sqrt{x^2+y^2}\,dA$ in polar coordinates. Do not evaluate the integral.

6. (10 points) Rewrite the following integral

$$\int_0^\pi \int_0^{\frac{\pi}{4}} \int_0^{3\sqrt{2}} \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta$$

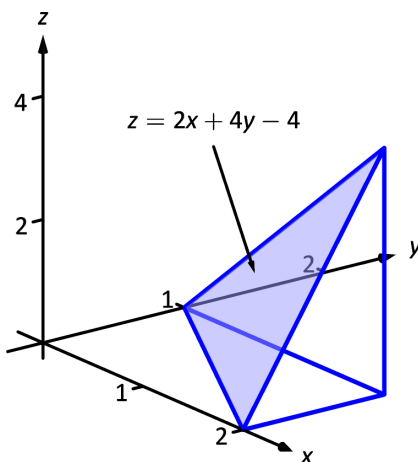in Cartesian coordinates in the order $dz\,dy\,dx$. Do not evaluate the integral.

7. (10 points) Let $\vec{F}(x,y) = \langle (1+xy)e^{xy},\ x^2 e^{xy}\rangle$ be a given force field and let

$$C: \vec{r}(t) = \langle \cos t,\ 2\sin t\rangle,\ \ 0 \le t \le 1$$

be a given path.

(a) (5 points) Find a potential function for $\vec{F}$.

(b) (5 points) Find $\displaystyle\int_C \vec{F}\cdot d\vec{r}$, the work done by $\vec{F}$ in moving an object along C.

8. (10 points) Let T be the tetrahedron pictured below, and let S be its closed boundary surface (consisting of all four faces) oriented outwards.
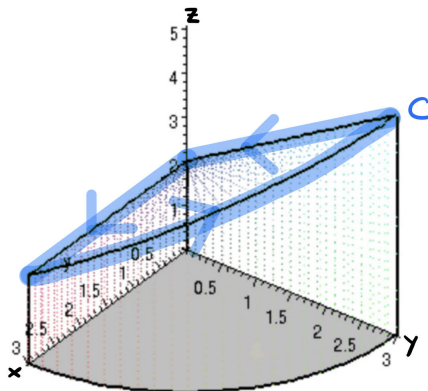

$$z = 2x + 4y - 4$$

Evaluate

$$\iint\limits_{S} \vec{F} \cdot d\vec{S},$$

where $\vec{F}(x, y, z) = \langle x + e^z, \ y + xz^2, \ xe^y \rangle$.

9. (10 points) Let the surface S be the portion of the plane $z = y + 2$ that lies within the cylinder $x^2 + y^2 = 9$ in the first octant. Let C be the boundary of S, where C is oriented counterclockwise when viewed from above. The contour C is shown in blue in the diagram below.

Evaluate $\displaystyle\int_{C} \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle e^{2x} + 2y, x + \cos y, \sin z \rangle$.

End of Math 226 Final Exam

The nine problems are spaced apart in the following pages for convenience.

1. (10 points) Find an equation for the plane in $\mathbb{R}^3$ that contains the points $(0, 1, 2)$ and $(1, 2, 1)$ and that is parallel to the line $\vec{r}(t) = \langle 1 + 2t, \ -t, \ -4 - t \rangle$.

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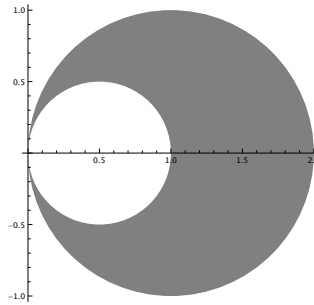
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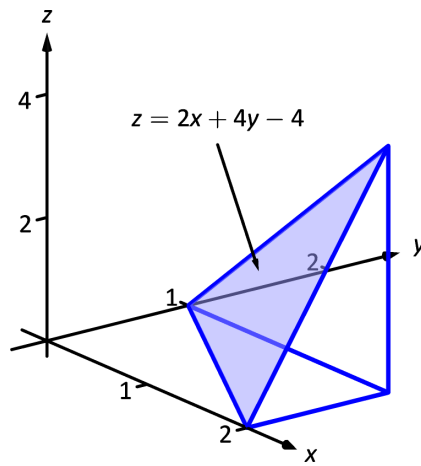
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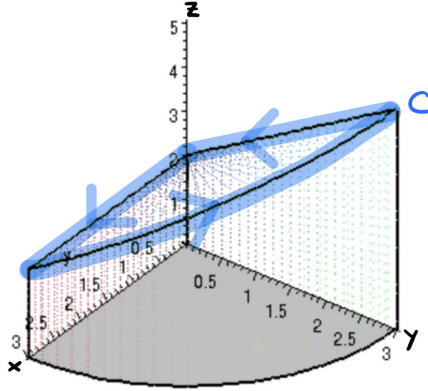

Evaluate

$$\iint_S \vec{F} \cdot d\vec{S},$$

where $\vec{F}(x, y, z) = \langle x + e^z, \ y + xz^2, \ xe^y \rangle$.

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Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle e^{2x} + 2y, x + \cos y, \sin z \rangle$.