**Math 126** **Final Exam** **Fall 2020**

11/18/2020 (online, submitted on Gradescope)

**Directions.** (Please read!)

- You must *show all your work and justify your methods* to obtain full credit. Name any theorems that you use. Clearly indicate your final answers.
- Simplify your answers to a reasonable degree.
- You may use your personal notes, your book and a calculator during the test. You can use your cell phone only at the end of the test to take pictures of your paper and submit. Collaboration with other students/external people is strictly forbidden.
- Your camera should be on and your microphone muted at all times during the test. If you experience a technical issue, please let your instructor know immediately. For any question to your instructor/TA, please use the chat.
- The test lasts *two hours*. You will have additional 20 minutes to submit on Gradescope. Please use the extra time exclusively for submission. Late submissions will be accepted only under exceptional circumstances.
- Please indicate the question number/part you are working on on your paper. Once you are done, you should submit one pdf on Gradescope. Please match the pages with the question numbers on the outline.
- Remember, USC considers cheating to be a serious offense; the minimum penalty is failure for the course. In the current circumstances, cheating includes communication with your peers via email, social media, etc.

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**Problem 1.** Find the following limit.

$$\lim_{x \to 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}}.$$

**Hint.** Use Taylor expansion.

**(10 pt.)**

**Problem 2.** Evaluate the following integrals.

*a)*

$$\int e^{1-ax} \sin(1+ax)\, dx; \qquad a \in \mathbb{R}$$

*b)*

$$\int \frac{x}{\sqrt{4x^2+4x+5}}\, dx$$

**(20 pt.)**

**Problem 3.** *a*) Compute the exact value of $\displaystyle\int_0^{\infty} \frac{4\,dx}{x^2+6x+8}$ if it is convergent. If it is divergent, explain why.

*b*) Decide whether the integral $\displaystyle\int_0^{\infty} \frac{e^{1-x}\,dx}{\sin(e^{-x})}$ converges or diverges, justifying carefully.

**(20 pt.)**

**Problem 4.** Let $R$ be the region bounded by the $y$-axis and the curves $y = x + 1$ and $y = x^3 + x$.

(a) Write down an integral giving the volume of the solid obtained by rotating $R$ around the $x$-axis. You do not have to evaluate it.

(b) Write down an integral giving the volume of the solid obtained by rotating $R$ around the line $x = 1$. You do not have to evaluate it.

**(20 pt.)**

**Problem 5.** The arc of the parabola $y = 2x^2$ between $x = 0$ and $x = 4$ is rotated about the $y$-axis. The obtained cup-like surface is filled with water. Find the work required to pump the water out from the top, assuming that $x$ and $y$ are measured in meters. (*Hint.* density of water $\rho = 1000\, kg/m^3$, $g = 9.8\, m/s^2$ )

**(15 pt.)**

**Problem 6.** Determine whether the given numerical series are convergent or divergent. Justify your answer in each case.

*a)* $\displaystyle\sum_{n=1}^{\infty} \frac{\ln^6 n}{n\sqrt{n}}$

*b)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \sin(1/n)}$

*c)* $\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{2} - \arctan n\right)$

**(30 pt.)**

**Problem 7.** Consider the power series

$$\sum_{n=1}^{\infty} \frac{(nr)^n}{n!}(x-1)^n,$$

where $r \geq 0$.

*a*) Find its radius of convergence as a function of $r$ (*Hint.* $lim_{n\to\infty}(1+1/n)^n = e$);

*b*) Find one value of $r$ such that the series converges at $x = 4$ and diverges at $x = -4$;

*c*) For what values of $r$ does the series converge for every real $x$?

**(15 pt.)**

**Problem 8.** Consider the following function: $f(x) = \dfrac{cos(x^3) - 1}{x^6}$.

*a*) Find a power series representation of $f(x)$ about $x = 0$. Write the power series in sigma notation, and write out the first three nonzero terms.

*b*) Using your answer from part a., find the number of nonzero terms in the series for $\int_0^{\pi/4} \frac{cos(x^3)-1}{x^6} dx$ that will guarantee an error of at most $\frac{1}{10^5}$.

**(15 pt.)**

**Problem 9.**

*a*) Find the area in the first two quadrants of the region bounded by the innermost loop of the parametric curve given by $x = cos(t), y = tsin(t)$ and $x = -1$ and $x = 1$.

*b*) Set up, but do not evaluate, the length of the curve bounding the top of the region in part a.

**(15 pt.)**