Math 125 - Fall 2018 | Wednesday, December 5th

Final Exam

Name: ______________________________ **ID#:** ______________________________

Please circle your section below:

Dreyer	Hogancamp 9am	Neshitov 12pm
Hall 9am	Hogancamp 10am	Neshitov 1pm
Hall 10am	Mancera	Yue
Hall 2pm	Mikulevicius	

Instructions

- You are allowed one page of notes (front and back) and a writing implement.
- All cell phones, calculators, and other devices must be turned off and out of reach.
- In all problems, you should include appropriate justification for your work. Solutions without much justification may not receive much credit.
- When asked to perform a calculation, put a box around your $\boxed{\text{final answer}}$.
- Answers do not need to be fully simplified, but special values of standard functions should be evaluated (e.g. $\sin(\frac{\pi}{2}) = 1$)

Problem	Score
1	/ 24
2	/ 24
3	/ 24
4	/ 16
5	/ 16

Problem	Score
6	/ 28
7	/ 16
8	/ 16
9	/ 16
10	/ 20
Total	/ 200

1. In parts (a)–(d), evaluate the given limit.

   (You may not appeal to l'Hôpital's rule.)

   (a) (6 points) $\displaystyle\lim_{x\to 2} \frac{2x^2 - 3x - 2}{x - 2}$

   (b) (6 points) $\displaystyle\lim_{x\to 0} \frac{x}{x + \sin(2x)}$

(c) (6 points) $\lim_{x \to \infty} \left(\sqrt{x^2 + 4x} - x\right)$

(d) (6 points) $\lim_{x \to 1} \frac{\ln(x) - \ln(1)}{x - 1}$

2. In parts (a)–(d), compute the derivative of the given function $f(x)$. Cite any theorems or rules of differentiation you use.

   (a) (6 points) $f(x) = \dfrac{\tan(x)}{x}$

   (b) (6 points) $f(x) = (x^2 + 1)^4(1 - x)^3$

(c) (6 points) $f(x) = (1+x)^{\frac{1}{x}}$

(d) (6 points) $f(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \frac{\theta}{\theta^4+1}\,\mathrm{d}\theta$

3. In parts (a)–(d), evaluate the given indefinite or definite integral.

(a) (6 points) $\displaystyle\int (x^5 + x^\pi + x^{1/2})\,\mathrm{d}x$

(b) (6 points) $\displaystyle\int_0^{\ln(2)} \frac{e^x}{e^x + 1}\,\mathrm{d}x$

(c) (6 points) $\displaystyle\int \frac{(1+x^{1/4})^{1/3}}{x^{3/4}}\,\mathrm{d}x$

(d) (6 points) $\displaystyle\int_{-1}^{1} \frac{x^3}{\cos(1+x^{10})}\,\mathrm{d}x$

4. Let $f(x) = \begin{cases} x^2 \cos(\frac{\pi}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a) (8 points) Show that $f(x)$ is continuous at $x = 0$.

$$\text{Function: } f(x) = \begin{cases} x^2\cos(\frac{\pi}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(b) (8 points) Is $f(x)$ is differentiable at $x = 0$?

5. (16 points) Consider the curve given by the equation

$$\sin(xy) = \cos(y) + x.$$

Find the tangent line to this curve at the point $(1, \pi)$, and use this to give an estimate of the y-value for a nearby point on the curve where $x = 0.98$.

(You may leave any constants in exact form, without using decimal approximations.)

(Extra space)

6. Let $f(x) = \dfrac{4x}{x^2+2}$. The first and second derivatives of $f(x)$ are:

$$f'(x) = \frac{8-4x^2}{(x^2+2)^2}, \qquad f''(x) = \frac{8x(x^2-6)}{(x^2+2)^3}.$$

(You do not need to justify these formulas.)

(a) (4 points) Give the largest domain on which $f(x)$ is continuous, with a brief explanation for your answer.

(b) (8 points) Find all intervals where $f(x)$ is increasing or decreasing, and indicate any local maximums or minimums.

$$f(x) = \frac{4x}{x^2+2}, \qquad f'(x) = \frac{8-4x^2}{(x^2+2)^2}, \qquad f''(x) = \frac{8x(x^2-6)}{(x^2+2)^3}.$$

(c) (8 points) Find all intervals where $f(x)$ is concave up or concave down, and indicate any inflection points.

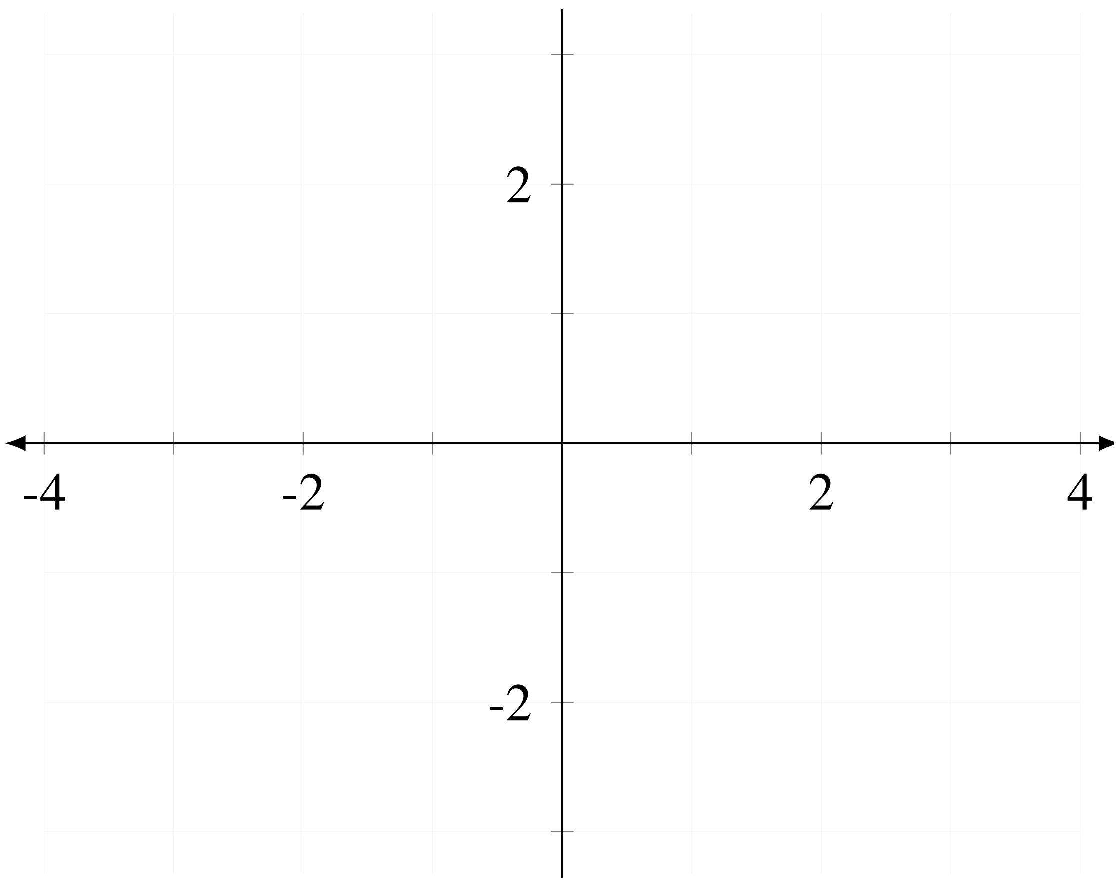
(d) (4 points) Find any horizontal or vertical asymptotes for the graph of $y = f(x)$. Be sure to justify your answer.

$$f(x) = \frac{4x}{x^2+2}, \qquad f'(x) = \frac{8-4x^2}{(x^2+2)^2}, \qquad f''(x) = \frac{8x(x^2-6)}{(x^2+2)^3}.$$

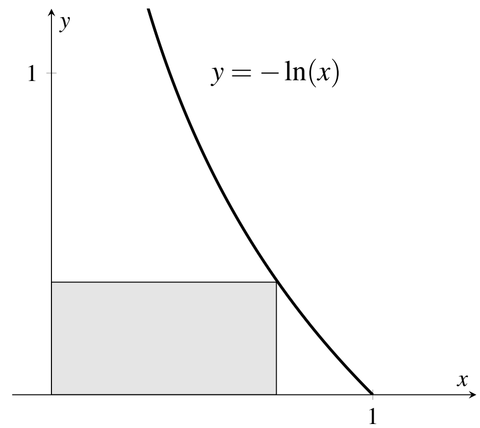
(e) (4 points) Using your answers to the previous parts, sketch the graph of $f(x)$ on the axes provided, ensuring accuracy of the following features as best you can:

- Intervals where $f(x)$ is increasing / decreasing
- Intervals where $f(x)$ is concave up / concave down
- Any local maximums or minimums, and inflection points

(The coordinates for local extrema and inflection points do not have to be precisely estimated, but should lie between the correct grid lines. For reference, $\sqrt{2} \approx 1.414$ and $\sqrt{6} \approx 2.449$.)



7. (16 points) What is the area of the largest rectangle that is contained in the region bounded by coordinate axes and the curve $y = -\ln(x)$ and has sides parallel to the coordinate axes? Be sure to justify that it is the largest.



8. Let $f(x) = x^4 + x - 3$.

   (a) (8 points) Show that $f(x)$ has a root in the interval $[-2, 0]$, and a root in the interval $[0, 2]$.

(b) (8 points) Show that $f(x)$ does not have more than two roots.

9. Let $f(x) = 2x + \sqrt{x^3 + 1}$.

(a) (8 points) Show that $f(x)$ is one-to-one on the domain $[-1, \infty)$ and find the domain of $f^{-1}(x)$.

(b) (8 points) Find the derivative of $f^{-1}(x)$ at $x = 1$. (Hint: Note that $f(0) = 1$.)

10. A deposit of ore contains 100-mg of radium-226, which undergoes radioactive decay. After 500 years, 80.4% of the original mass of radium-226 remains.[1]

(a) (8 points) Find the mass $m(t)$ of radium-226 that remains after t years.

(b) (6 points) What is the half-life of radium-226?

[1] Constants in your answers to this problem may be left in exact form, e.g. $(1.234) \cdot \ln(5.678)$.

(c) (6 points) When will there be 20-mg of radium-226 remaining?