The University of Southern California

MATH 125 Fall 2020

Final Exam

Instructions: Please show all of your work and reasoning.

In the final answers, keep the irrational numbers such as $\pi, e, \ln 2, \sqrt{2}$, do not convert ordinary fractions to decimals, and do not approximate anything.

If you have a question, please write to the instructor using the private chat function of the zoom meeting. Other than that, you may not communicate with anybody during the exam.

The exam lasts two hours (120 minutes).

Problem 1 [5 points]: Write the following limit as a definite integral. You may use the properties of logarithms to help with your algebra. Do not evaluate the integral.

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\left(1+\frac{1}{n}\right)^3 \left(1+\frac{2}{n}\right)^3 \cdots \left(1+\frac{n}{n}\right)^3\right).$$

Problem 2 [6 points]: Determine the value of the real number A so that a finite limit exists, and then compute the limit:

$$\lim_{x\to+\infty} \left(\sqrt{Ax^2+2x} - 5x\right).$$

Problem 3 [5 points]: Use the Squeeze Theorem to find the limit

$$\lim_{x\to 0+} \frac{\ln(1+x)-x}{x^{3/2}}.$$

You may assume that the following inequalities are true for all $x > 0$:

$$\frac{x}{1+x} < \ln(1+x) < x.$$

Problem 4 [12 points]: Consider the following function, defined on $[-2,2]$:

$$f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1, \\ \\ \dfrac{x^2 - x}{2} + 1, & -1 < x \le 2. \end{cases}$$

(a) Find the global (absolute) **minimum** value of $f(x)$ on $[-2,2]$, and the point where it occurs. If $f(x)$ has no global minimum, explain why.

(b) Next, we consider the above function $f = f(x)$ on a *smaller* interval $\mathbf{[-2,0]}$.
   (i) Are the hypotheses of the Mean Value Theorem satisfied by the function f on the interval $[-2,0]$?
   (ii) Does the conclusion of the Mean Value Theorem hold for the function f on the interval $[-2,0]$?

Problem 5 [8 points]: Let $F = F(x)$ be the function

$$F(x) = \int_1^{x^3} \sqrt{t^2+1}\, dt, \ -\infty < x < +\infty.$$

(a) Explain why the function F is invertible.

(b) Compute $(F^{-1})'(0)$.

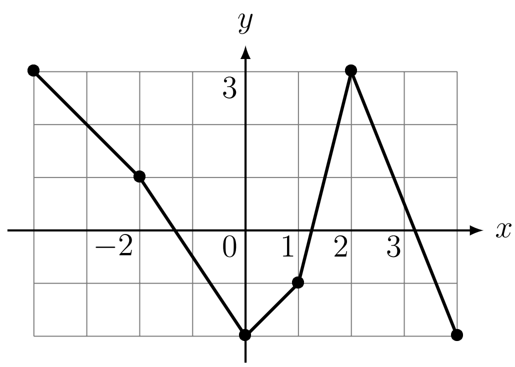
Problem 6 [20 points]:

(a) Let $f(x) = \dfrac{\cos(\pi x^{-2})}{x^3}$. Compute the **average value** of f on the interval $\left[1, \sqrt{2}\right]$.

(b) Evaluate the following integral by making the substitution $u = x^2$ and then interpreting the result as an area:

$$\int_0^1 x\sqrt{2x^2 - x^4}\, dx.$$

Problem 7 [24 points]: Consider the following graph:



(a) Suppose that this is the graph of a function $f = f(x)$. In particular, $f(-1) = -1/2$.

(i) Consider the function $g(x) = f(3x - 4)$.
Compute $g'(2)$, if possible. If this is not possible, then explain why.

(ii) Consider the function $h(x) = f\big(f(x)\big) + \ln\big(1 - \tan(\pi f(x)/2)\big)$.
Compute $h'(-1)$, if possible. If this is not possible, then explain why.

(b) Now suppose that this is the graph of the **derivative** u' of a function $u = u(x)$, and suppose moreover that $u(-2) = 0$.

(i) Compute an approximate value of $u(-2.1)$. Show your work.

(ii) For each of the following statements, indicate whether it is true, false, or you need more information to make the conclusion, and give a brief explanation.

u is concave up on $(0, 2)$	**True**	**False**	**Need More Info**
u has a local maximum on $[2, 4]$	**True**	**False**	**Need More Info**

Problem 8 [9 points]: The radius r of a right circular cone is *increasing* at the rate of 1 foot per second, and the height h of the cone is *decreasing* at the rate of 2 feet per second.

At a particular time, the radius of the cone is 30 feet and the height of the cone is 20 feet. Is the volume of the cone increasing or decreasing at that moment? Explain your conclusion.

Problem 9 [7 points]: Does the curve $e^y = x + y$ have any points at which the tangent line to the curve is horizontal? Explain your conclusion.

Problem 10 [12 points]: A clothing store produces and sells suits. The total (cumulative) cost, in dollars, of producing q suits is

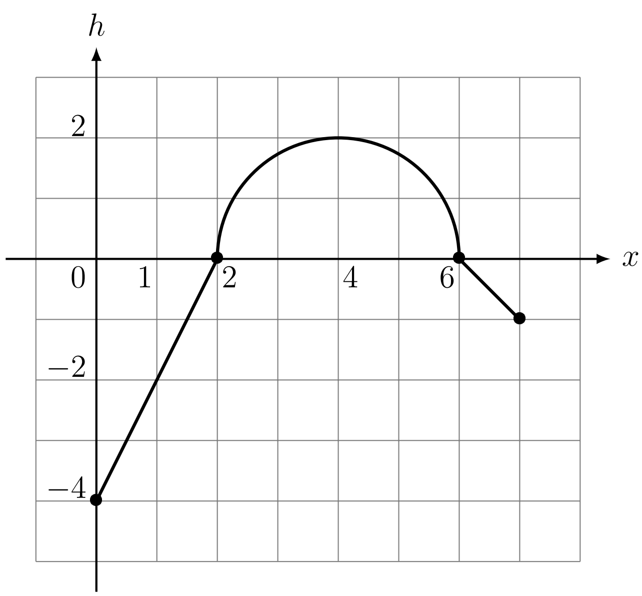
$$C(q) = 4000 + 0.25q^2.$$

To sell q suits, the price per suit, in dollars, must be

$$p = 150 - 0.5q.$$

(a) What is the profit from selling q suits?
(b) How many suits must the store produce and sell to maximize profit?
(c) What is this maximum profit?
(d) What price per suit must the store charge to maximize profit?

Problem 11 [9 points]: The graph of the function $h = h(x)$ consists of two straight lines and a semicircle, shown below.



Use the picture to evaluate the following two integrals:

$$\int_1^4 h(x)\,dx, \qquad \int_0^7 h(x)\,dx.$$