Assignment 4

(1) When X has a Binomial $\mathrm{Bin}(n,p)$ distribution with $0 < p < 1$, we say that $g(p) = p/(1-p)$ is the 'odds ratio'. For instance, when $p = 2/3$ the odds ratio is 2, corresponding to 'odds' of '2 to 1' for success. Show that there is no unbiased estimator for the odds $g(p)$. Hint: First consider the identity

$$E_p[\theta(X)] = \frac{p}{1-p}$$

in the case $n = 1$.

(2) Let $X \sim \mathcal{P}(\lambda)$ be an observation from the Poisson distribution satisfying

$$P_\lambda(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}. \tag{1}$$

(For instance, X might be the number of arrivals in the time interval $[0, 1]$, when arrivals follow a Poisson process with rate λ.)

(a) Find an unbiased estimate of $\phi(\lambda) = e^{-3\lambda}$ (the probability that there are no arrivals in the interval $[1, 4]$). Hint: Find a function g that satisfies

$$E_\lambda[g(X)] = e^{-3\lambda}$$

using (1), the infinite series representation of the exponential function and the fact that $3 = 1 + 2$.

(b) Compute the variance of the resulting estimator.

(c) Are there any other unbiased estimators of $g(\lambda)$. Is the unbiased estimator that was derived in part a) UMVU?

(d) Compute the value of the estimator for some small values of X, and comment on any peculiarities you observe.

(e) What types of inferential or statistical conclusions can be drawn from this example?

(3) For $\theta \in \mathbb{R}$, let $X_1, \ldots, X_n$ be independent random variables with the Cauchy distribution having density

$$p(x, \theta) = \frac{1}{\pi(1 + (x-\theta)^2)} \quad x \in \mathbb{R}.$$

(a) Compute the Fisher information for the location parameter θ in one observation, and for the entire sample.

(b) Determine the asymptotic distribution of a consistent maximum likelihood estimator of θ, properly centered and scaled, via Cramér's theorem.

(c) Write out the equation one must solve in order to determine the maximum likelihood, in particular in the case of a large sample, and comment on any potential difficulties.

(4) Let $X_1, \ldots, X_n$ be independent random variables with density

$$p(x;\theta) = \exp\left\{-(x-\theta) - e^{-(x-\theta)}\right\}, \qquad x \in \mathbb{R}, \quad \theta \in \mathbb{R}.$$

(a) If X has density $p(x;\theta)$, use a change of variables to show that

$$Y = e^{-(X-\theta)} \quad \text{has distribution} \quad Y \sim \mathrm{Exp}(1).$$

(b) Compute the maximum likelihood estimator of θ, and show that it is the unique maximizer of the likelihood.

(c) Compute the Fisher information for θ in one observation and in the entire sample. Use the asymptotic normality theorem for maximum likelihood estimators to state the limiting distribution of the maximum likelihood estimator, properly centered and scaled.

(5) Prove the multivariate information bound when $\theta \in \mathbb{R}^p$ and $E_\theta[T(\boldsymbol{X})] = g(\theta) \in \mathbb{R}^q$. Hint: compute the covariance matrix of $T(\boldsymbol{X})' - \dot{g}(\theta)'\boldsymbol{I}(\theta)^{-1}U(\theta, \boldsymbol{X})$).

(6) Compute the information matrix for the vector μ contained in one observation from a multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$ with non-singular covariance matrix Σ.

(7) Compute the Information Inequality (Cramér Rao) lower bound for the "signal to noise" ratio μ/σ when observing n independent normal observations with distribution $\mathcal{N}(\mu, \sigma^2)$.

(8) Find example from the families of classical distributions, (such as the Poisson, Gamma, Beta, and Binomial) for which one directly apply the central limit theorem to verify that the maximum likelihood estimate, centered and properly scaled, converges in distribution to a mean zero normal distribution with variance (or covariance matrix) given by the inverse Information, that is, without the need to invoke Cramér's theorem?

(9) For the maximum likelihood estimate $X_{(n)}$ of θ from an independent sample of size n from the $U[0,\theta]$ distribution, show that $n(\theta - X_{(n)})$ converges to a non-trivial distribution E, and find the density function of the limit E. Compare this rate of convegence to the parametric rate given by Cramér's theorem.

(10) For $r \geq 1$ let $X_1, \ldots, X_n$ be given by

$$X_i = \theta(1 - U_i^r), \quad i = 1, \ldots, n$$

where $U_1, \ldots, U_n$ are independent $U[0,1]$ random variables, and again let $X_{(n)}$ be the maximum observation. Determine a scaling $s_{\theta,n}$ that may depend on θ and n, and a non-trivial distribution D such that

$$s_{\theta,n}(\theta - M_n) \to_d D,$$

where $\to_d$ denotes convergence in distribution, and verify that the case $r = 1$ agrees with the solution of the previous problem.

(11) Show that, when integration and differentiation can be interchanged, we have

$$-E\left[\frac{\partial^2}{\partial\theta^2} \log p(\mathbf{X}, \theta)\right] = I_{\boldsymbol{X}}[\theta]$$