EXERCISE SHEET 5

The exercise numeration aligns with the numbering system used in Chapter 5, Chapter 6, Chapter 7 and Chapter 9 in the book "Introduction to Probability" by David Anderson, Timo Seppalainen, and Benedek Valko.

1. Exercises from Chapter 5

Ex. 2: Suppose that the discrete variable X has moment generating function

$$M_X(t) = \frac{1}{2} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t}.$$

(a) Find $\mathbb{E}[X]$.
(b) Find the probability mass function.

Ex. 5: The moment generating function of the random variable X is $M_X(t) = e^{3(e^t-1)}$. Find $\mathbb{P}(X = 4)$.
Hint: Recall the moment generating function of a Poisson random variable.

Ex. 7: Suppose $X \sim \text{Exp}(\lambda)$ and $Y = \ln(X)$. Find the probability density of Y.

Ex. 9: Let $X \sim \text{Bin}(n, p)$.
(a) Find the moment generating function $M_X(t)$.
(b) Use part (a) to find $\mathbb{E}[X]$, $\mathbb{E}[X^2]$ and $\text{Var}(X)$.

Ex. 10: Suppose that X has moment generating function

$$M(t) = \left(\frac{1}{5} + \frac{4}{5}e^{-t}\right)^{30}.$$

What is the distribution of X?

Ex. 16: Let $X \sim \text{Unif}[0, 1]$.
(a) Compute $\mathbb{E}[X^n] = \int_0^1 x^n\, dx$.
(b) Compute $M_X(t)$. Find the Taylor series expansion of $M_X(t)$ and identify the coefficients.

Ex. 18: Let $X \sim \text{Geom}(p)$.
(a) Compute the moment generating function $M_X(t)$ of X. Be careful about the possibility that $M_X(t)$ might be infinite.
(b) Use the moment generating function to compute the mean and the variance of X.

Ex. 20: Suppose that the random variable X has density function $f(x) = \frac{1}{2}e^{-|x|}$.
(a) Compute the moment generating function $M_X(t)$ of X. Be careful about the possibility that $M_X(t)$ might be infinite.
(b) Use the moment generating function to compute the n-th moment of X.

Ex. 21: Let $Y = aX + b$ where $a, b \in \mathbb{R}$. Express the moment generating function $M_Y(t)$ in terms of $M_X(t)$.

Ex. 22: Let $X \sim \text{Exp}(\lambda)$. Find the moment generating function of $Y = 3X - 2$.

Ex. 24: Let $X \sim \mathcal{N}(0, 1)$ and $Y = e^X$. The random variable Y is called *log-normal random variable*. Find the probability density function of Y.

Ex. 28: Let $X \sim \text{Unif}[-1, 2]$. Find the probability density function of $Y = X^4$.

Ex. 31: Suppose $U \sim \text{Unif}[0, 1]$. Let $Y = e^{\frac{U}{1-U}}$. Find the probability density function of Y.

2. Exercises from Chapter 6

Ex. 7: Consider the triangle with vertices (0,0), (1,0), (0,1). Suppose that (X, Y) is a uniformly chosen random point from this triangle.
(a) Find the marginal density functions of X and Y.
(b) Calculate the expectations $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
(c) Calculate $\mathbb{E}[XY]$.

Ex. 13: Consider the disk with radius r_0 centered at $(0, 0)$ and consider a point (X, Y) chosen uniformly at random from such a disk. Determine whether random variables X and Y are independent or not.

Ex. 15: Let Z, W be independent standard normal random variables and $-1 < \rho < 1$. Check that if $X = Z$ and $Y = \rho Z + \sqrt{1 - \rho^2} W$, then the pair (X, Y) has a standard bivariate normal distribution with parameter ρ.

Ex. 18: Suppose that X and Y are integer-valued random variables with joint probability mass function given by

$$p_{X,Y}(a, b) = \begin{cases} \frac{1}{4a}, & \text{for } 1 \le b \le a \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that this is indeed a joint probability mass function.
(b) Find the marginal probability mass functions of X and Y.
(c) Find $\mathbb{P}(X = Y + 1)$

Ex. 19: The joint probability mass function of the random variables (X, Y) is given by

$$p_{X,Y}(0, 0) = p_{X,Y}(0, 1) = \frac{1}{18}, \qquad p_{X,Y}(1, 1) = p_{X,Y}(1, 2) = \frac{5}{18},$$
$$p_{X,Y}(1, 0) = \frac{2}{18}, \qquad p_{X,Y}(0, 2) = \frac{4}{18}.$$

(a) Find the marginal probability mass function of X and Y.
(b) Suppose Z and W are independent random variables and that Z and X are equal to X and Y, respectively. Give the joint probability mass function $p_{Z,W}$ of (Z, W).

Ex. 20: Let $(X_1, X_2, X_3, X_4) \sim \text{Multinomial}(n, 4, \frac{1}{6}, \frac{1}{3}, \frac{1}{8}, \frac{3}{8})$. Derive the joint probability mass function of X_3, X_4.

Ex. 35: Suppose that X and Y are random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4}(x + y)\,, & 0 \le x \le y \le 2\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

(a) Check that $f_{X,Y}$ is a joint density function.
(b) Calculate the probability $\mathbb{P}(Y < 2X)$.
(c) Find the marginal density function $f_Y(y)$ of Y.

Ex. 47: Let $X_1, \dots, X_n$ be independent random variables with the same cumulative distribution function F. Let us assume that F is continuous. Define

$$Z = \min\{X_1, \dots, X_n\}\,, \qquad W = \max\{X_1, \dots, X_n\}\,.$$

(a) Find the cumulative distribution functions F_Z of Z and F_W of W.
(b) Assume additionally that the random variables $X_1, \dots, X_n$ are continuous and have same probability density function f. Find the probability density function f_Z of Z and f_W of W.
(c) Answer to (a) and (b) assuming that $X_1, \dots, X_n$ are i.i.d. and $X_1 \sim \text{Unif}[0, 1]$.

3. Exercises from Chapter 7

Ex. 4: Suppose that X and Y are independent exponential random variables with parameters $\lambda \neq \mu$. Find the density function of $X + Y$.

Ex. 15: Let X be an integer chosen uniformly at random from the set $\{1, \dots, n\}$ and let Y be an integer chosen uniformly at random from the set $\{1, \dots, m\}$. Find the probability mass function of $X + Y$.

4. Exercises from Chapter 9

Ex. 2: Let $X \sim \text{Exp}(1/2)$.
(a) Use Markov's inequality to find an upper bound for $\mathbb{P}(X \ge 6)$.
(b) Use Chebyshev's inequality to find an upper bound for $\mathbb{P}(X \ge 6)$.

Ex. 17: Let $X \sim \text{Poi}(100)$.
(a) Use Markov's inequality to find an upper bound for $\mathbb{P}(X \ge 120)$.
(b) Use Chebyshev's inequality to find an upper bound for $\mathbb{P}(X \ge 120)$.
(c) Using the fact that, if $X_1, \dots, X_n$ are i.i.d. with $X_1 \sim \text{Poi}(1)$, then $X_1 + \ldots + X_n \sim \text{Poi}(n)$, use the Central Limit Theorem to approximate the value $\mathbb{P}(X > 120)$.

Ex. 21: Let $X_1, \dots, X_{500}$ be i.i.d. random variables with expected value 2 and variance 3. The random variables $Y_1, \dots, Y_{500}$ are independent of $X_1, \dots, X_{500}$, also i.i.d., but they have expected value 2 and variance 2. Use the Central Limit Theorem to estimate $\mathbb{P}(\sum_{i=1}^{500} X_i > \sum_{i=1}^{500} Y_i + 50)$.

Hint: Use the Central Limit Theorem for the random variables $X_1 - Y_1, X_2 - Y_2, \ldots$.