EXERCISE SHEET 4

The exercise numeration aligns with the numbering system used in Chapter 4 in the book "Introduction to Probability" by David Anderson, Timo Seppalainen, and Benedek Valko.

Exercises with ** are harder and the solution is not provided. Discussions on these exercises is possible if there is time during the lectures or by asking an office hour.
For exercise with *, the solution is not provided.

1. Exercises

Ex. 3: Approximate the probability that out of 300 die rolls we get exactly 100 numbers that are multiples of 3.

Ex. 5: Liz is standing on the real number line at position 0. She rolls a die repeatedly. If the roll is 1 or 2, she takes one step to the right. If the roll is 3, 4, 5 or 6, she takes two steps to the right. Let X_n be Liz's position after n rolls of the die.
(a) Find $\lim_{n\to\infty} \mathbb{P}(X_n > 1.6n)$.
(b) Find $\lim_{n\to\infty} \mathbb{P}(X_n > 1.7n)$.

Ex. 11: On the first 300 pages of a book, you notice that there are, on average, 6 typos per page. What is the probability there there will be at least 4 typos on page 301? State clearly the assumptions you are making.

Ex. 12: Let $T \sim \text{Exp}(\lambda)$. Compute $\mathbb{E}[T^3]$.

Ex. 15: Suppose that a class of students is star-gazing on top of the local mathematics building from 11pm to 3am. Suppose further that meteors arrive (i.e. they are seen) according to a Poisson process with intensity $\lambda = 4$ per hour.
(a) Find the probability that the students see more than 2 meteors in the first hour.
(b) Find the probability that the students see zero meteors in the first hour, but at least 10 meteors in the final three hours (from midnight to 3am).
(c) Given that there were 13 meteors seen all night, what is the probability that there were no meteors seen in the first hour?

Ex. 20: You flip a fair coin 10,000 times. Approximate the probability that the difference between the number of heads and number of tails is at most 100.

Ex. 23: Suppose that the distribution of the lifetime of a car battery, produced by a certain car company, is well approximated by a normal distribution with a mean of $1.2 \cdot 10^3$ hours and variance 10^4. What is the approximate probability that a batch of 100 car batteries will contain at least 20 whose lifetimes are less than 1100 hours?

Ex. 28: For which values of p will the probability mass function of the $\mathrm{Bin}(n, p)$ distribution have its maximum at n?

Ex. 32: Recall that, if $X \sim \mathrm{Poi}(\lambda)$ then $\mathbb{E}[(X)_n] = \lambda^n$ (see third problem of the additional exercises for the sheet n.3). Use this formula to compute $\mathbb{E}[X^3]$.

Ex. 35: Every morning Jack flips a fair coin ten times. He does this for an entire year. Let X denote the number of days when all the flips come out the same way (all heads or all tails).

(a) Give the precise expression for the probability $\mathbb{P}(X > 1)$.

(b) Apply either the normal or the Poisson approximation to give a simple estimate for $\mathbb{P}(X > 1)$. Explain your choice of approximation.

Ex. 36:** How many randomly chosen guests should I invite to my party so that the probability of having a guest with the same birthday as mine is at least $\frac{2}{3}$?

Ex. 41: We roll a die 72 times. Approximate the probability of getting exactly 3 sixes with both the normal and the Poisson approximation and compare the results with the exact probability 0.000949681.

Ex. 44: Suppose that 50% of all watches produced by a certain factory are defective (the other 50% are fine). A store buys a box with 400 watches produced by this factory. Assume this is a random sample from the factory.

(a) Write an expression for the exact probability that at least 215 of the 400 watches are defective.

(b) Approximate the probability, using either the Poisson or normal approximation, whichever is appropriate, that at least 215 of the 400 watches are defective.

Ex. 49*: Suppose that you own a store that sells a particular stove for \$1000. You purchase the stoves from the distributor for \$800 each. You believe that this stove has a lifetime which can be faithfully modeled as an exponential random variables with a parameter of $\lambda = 1/10$, where the units of time are years. You would like to offer the following extended warranty on this stove: if the stove breaks within r years, you will replace the stove completely (at a cost of \$800 to you). If the stove leasts longer than r years, the extended warranty pays nothing. Let \$$C$ be the cost you will charge the consumer for this extended warranty. For what pairs (C, r) will the expected profit you get from this warranty be zero? What do you think are reasonable choices for C and r? Why?

Ex. 50*: Suppose an alarm clock has been set to ring after T hours, where $T \sim \mathrm{Exp}(1/3)$. Suppose further that your friend has been starting at the clock for exactly 7 hours and can confirm that is has not yet rung. At this point, your friend wants to know when the clock will finally ring. Calculate her conditional probability that she needs to wait at least 3 more hours, given that she has already waited 7 hours. More generally, calculate the conditional probability that she needs to wait at least x more hours, given that she has already waited 7 hours.

Ex. 52*: Show that $\Gamma(r+1) = r\Gamma(r)$ for any $r > 0$. Deduce then that for any integer n, $\Gamma(n) = (n-1)!$.

Ex. 53*: Let $X \sim \Gamma(r, \lambda)$. Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Ex. 54:** The point of this exercise is to check that it is immaterial whether the inequality is $<$ or $\leq$ in the (weak) law of large numbers. Consider these two statements:

$$\forall \varepsilon > 0 \qquad \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| \leq \varepsilon\right) = 1\,, \tag{1}$$

and

$$\forall \varepsilon > 0 \qquad \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = 1\,. \tag{2}$$

(a) Show that (1) implies (2).
(b) Show that (2) implies (1).

Ex. 56:** Suppose that $X \sim \mathrm{Poi}(\lambda)$. Find the probability $\mathbb{P}(X \text{ is even})$. Your answer should not be written as an infinite series.

2. Solutions

Before starting with the solution of the exercises let us recall two important facts.

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then
$$X = \sigma Z + \mu$$
where $Z \sim \mathcal{N}(0,1)$. Hence
$$\mathbb{P}(X \leq t) = \mathbb{P}(\sigma Z + \mu \leq t) = \mathbb{P}\left(Z \leq \frac{t-\mu}{\sigma}\right) = \Phi\left(Z \leq \frac{t-\mu}{\sigma}\right),$$
where $\Phi(t) = \mathbb{P}(Z \leq t)$ is the cumulative distribution function of a standard normal random variable.
- For any $x \in \mathbb{R}$
$$\Phi(-x) = 1 - \Phi(x).$$
- **Central Limit Theorem (CLT)**: Consider $\{X_n\}_{n\geq 1}$ i.i.d. random variables with $\mathbb{E}[X_1] < \infty$ and $\mathrm{Var}(X_1) < \infty$. Define $S_n = \sum_{i=1}^n X_i$. Then
$$\mathbb{P}\left(\frac{S_n - n\mathbb{E}[X_1]}{\sqrt{n\,\mathrm{Var}(X_1)}} \leq t\right) \underset{n\to\infty}{\longrightarrow} \Phi(t).$$
So when n is large we can approximate $\mathbb{P}(S_n \leq t)$ with
$$\mathbb{P}(S_n \leq t) = \mathbb{P}\left(\frac{S_n - n\mathbb{E}[X_1]}{\sqrt{n\,\mathrm{Var}(X_1)}} \leq \frac{t - n\mathbb{E}[X_1]}{\sqrt{n\,\mathrm{Var}(X_1)}}\right) \underset{\text{CLT}}{\approx} \Phi\left(\frac{t - n\mathbb{E}[X_1]}{\sqrt{n\,\mathrm{Var}(X_1)}}\right).$$
The values of Φ can be found on the **gaussian table**.
- Fixed an integer t, if we want to use the CLT to approximate $\mathbb{P}(S_n = t)$ for n large, we can use the **continuity correction**, that is
$$\mathbb{P}(S_n = t) = \mathbb{P}\left(t - \frac{1}{2} \leq S_n \leq t + \frac{1}{2}\right),$$
since t is the only integer in the interval $\left[t - \frac{1}{2}, t + \frac{1}{2}\right]$.
- If $X \sim \mathrm{Ber}(p)$, then $\mathbb{E}[X] = p$ and $\mathrm{Var}(X) = p(1-p)$.
- Suppose to have n i.i.d. experiments, each one with probability of success equal to p, where p is small and n is large. Denote by X the number of successes. Then $X \sim \mathrm{Bin}(n,p)$. By the way, computations with such a distribution may be hard since n is big. So we can use the **Poisson approximation** to approximate X with a random variable $Y \sim \mathrm{Poi}(\lambda)$, where $\lambda = np$ and for $k \in \mathbb{N}$ (included $k = 0$) we have
$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}.$$
For Y we know that $\mathbb{E}[Y] = \mathrm{Var}(Y) = \lambda$.

Now we can start with the solutions of the exercises.

Ex. 3: For $i = 1, \dots, 300$ define the random variable
$$X_i = \begin{cases} 1, & \text{if the } i\text{-th roll gives a multiple of 3} \\ 0, & \text{otherwise.} \end{cases}$$

Define $S_{300} = \sum_{i=1}^{300} X_i$. We would like to use the CLT to approximate $\mathbb{P}(S_{300} = 100)$. We have to use the continuity correction. So

$$\mathbb{P}(S_{300} = 100) = \mathbb{P}(99.5 \le S_{300} \le 100.5) = \mathbb{P}(S_n \le 100.5) - \mathbb{P}(S_n \le 99.5) \approx$$
$$\underset{\text{CLT}}{\approx} \Phi\left(\frac{100,5 - 300 \cdot \mathbb{E}[X_1]}{\sqrt{300 \cdot \mathrm{Var}(X_1)}}\right) - \Phi\left(\frac{99,5 - 300 \cdot \mathbb{E}[X_1]}{\sqrt{300 \cdot \mathrm{Var}(X_1)}}\right).$$

Note that $X_1 \sim \mathrm{Ber}(1/3)$ and hence $\mathbb{E}[X] = 1/3$ and $\mathrm{Var}(X) = 2/9$. So we have

$$\Phi\left(\frac{100,5 - 300 \cdot \mathbb{E}[X_1]}{\sqrt{300 \cdot \mathrm{Var}(X_1)}}\right) - \Phi\left(\frac{99,5 - 300 \cdot \mathbb{E}[X_1]}{\sqrt{300 \cdot \mathrm{Var}(X_1)}}\right) = \Phi\left(\frac{0.5}{\frac{10\sqrt{3}}{3}}\right) - \Phi\left(\frac{-0.5}{\frac{10\sqrt{3}}{3}}\right) =$$
$$= \Phi\left(\frac{0.5}{\frac{10\sqrt{3}}{3}}\right) - \left[1 - \Phi\left(\frac{0.5}{\frac{10\sqrt{3}}{3}}\right)\right] = 2\Phi\left(\frac{0.5}{\frac{10\sqrt{3}}{3}}\right) - 1 \approx$$
$$\approx 2\Phi(0.09) - 1 \approx 2 \cdot 0.54 - 1 = 0.08\,.$$

Ex. 5: For $i \in \mathbb{N}$ define the random variable

$$Y_i = \begin{cases} 1\,, & \text{if the } i\text{-th outcome is 1 or 2} \\ 2\,, & \text{otherwise.} \end{cases}$$

Then $X_n = \sum_{i=1}^{n} Y_i$. Note that

$$\mathbb{E}[Y_1] = 1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{5}{3}\,,$$
$$\mathbb{E}[Y_1^2] = 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{2}{3} = 3\,,$$

and

$$\mathrm{Var}(Y_1) = \mathbb{E}[Y_1^2] - \mathbb{E}[Y_1]^2 = 3 - \frac{25}{9} = \frac{2}{9}\,.$$

We would like to use the CLT to approximate $\mathbb{P}(X_n > tn)$, where $t = 1.6$ (ex. (a)) or $t = 1.7$ (ex. (b)). So

$$\mathbb{P}(X_n > tn) = \mathbb{P}\left(\frac{X_n - n \cdot 5/3}{\sqrt{n \cdot 2/9}} > \frac{tn - n \cdot 5/3}{\sqrt{n \cdot 2/9}}\right) \underset{\text{CLT}}{\approx} 1 - \Phi\left(\frac{\sqrt{n}(t - 5/3)}{\sqrt{2/9}}\right).$$

Note that $1.6 < 5/3 < 1.7$. Recalling that, being $\Phi(x)$ a cumulative distribution function, then

$$\lim_{x\to+\infty} \Phi(x) = 1\,, \qquad \lim_{x\to-\infty} \Phi(x) = 0\,.$$

So

$$\lim_{n\to+\infty} \mathbb{P}(X_n > tn) = \lim_{n\to+\infty}\left[1 - \Phi\left(\frac{\sqrt{n}(t - 5/3)}{\sqrt{2/9}}\right)\right] = \begin{cases} 1 - 0 = 1\,, & \text{if } t = 1.6\,, \\ 1 - 1 = 0\,, & \text{if } t = 1.7\,. \end{cases}$$

Ex. 11: Let n be the number of characters in a page and let p be the probability that a single character is wrong (i.e. there is a typo). Note that n is large and p is small. Let X count the number of typos in a page. Since each character can be considered as an independent experiment with success probability p, we have that $X \sim \mathrm{Bin}(n,p)$. Since n is large and p is small, we can use the Poisson approximation

$$X \approx Y \sim \mathrm{Poi}(\lambda)\,,$$

where $\lambda = np = \mathbb{E}[X] = \mathbb{E}[Y] = 6$.
Since we have to compute $\mathbb{P}(X \geq 4)$, we have

$$\begin{aligned}\mathbb{P}(X \geq 4) &= 1 - \mathbb{P}(X < 4) = 1 - \sum_{i=0}^{3} \mathbb{P}(X = k) = \\ &= 1 - (e^{-6} + 6e^{-6} + 18e^{-6} + 36e^{-6}) = 1 - 61e^{-6}\,.\end{aligned}$$

Ex. 12: Recall that, if $T \sim \mathrm{Exp}(\lambda)$, then T has probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}\,, & \text{if } x \geq 0\,, \\ 0\,, \text{otherwise.} \end{cases}$$

Moreover $\mathbb{E}[T] = \frac{1}{\lambda}$ and $\mathrm{Var}(T^2) = \frac{1}{\lambda^2}$. Applying integration by parts we have

$$\begin{aligned}\mathbb{E}[T^3] &= \int_{-\infty}^{+\infty} x^3 f(x)\,dx = \int_0^{+\infty} x^3 \cdot \lambda e^{-\lambda x}\,dx = \\ &= [x^3(-e^{-\lambda x})]_0^{+\infty} - \int_0^{+\infty} 3x^2(-e^{-\lambda x})\,dx = \\ &= \frac{3}{\lambda}\int_0^{+\infty} x^2 \cdot \lambda e^{-\lambda x}\,dx = \frac{3}{\lambda}\int_{-\infty}^{+\infty} x^2 f(x)\,dx = \frac{3}{\lambda}\mathbb{E}[T^2] = \\ &= \frac{3}{\lambda}(\mathrm{Var}(T^2) + \mathbb{E}[T]^2) = \frac{3}{\lambda}\cdot\frac{2}{\lambda^2} = \frac{6}{\lambda^3}\,.\end{aligned}$$

Ex. 15: Denote by $N([t,s])$ the number of arrivals between time t and time s. Then $N([t,s]) \sim \mathrm{Poi}(\lambda(s-t))$ with $\lambda = 4$. Moreover $N([t,s])$ and $N([k,j])$ are independent if $(t,s) \cap (k,j) = \emptyset$.
Since we are looking from 11pm to 3am, we will think the time variable t to be a number in $[0,4]$, where $t = 0$ means 11 pm and $t = 4$ means 3am.
(a) We have

$$\mathbb{P}(N([0,1]) \geq 2) = 1 - [\mathbb{P}(N([0,1]) = 0) + \mathbb{P}(N([0,1]) = 1)] = 1 - [e^{-4} + 4e^{-4}] = 1 - 5e^{-4}\,.$$

(b) Since the arrivals are independent and uniform in time, we have

$$\begin{aligned}\mathbb{P}(N([0,1]) = 0, N([1,4]) \geq 10) &= \mathbb{P}(N([0,1]) = 0)\mathbb{P}(N([1,4]) \geq 10) = \\ &= e^{-4} \cdot \left[1 - \sum_{k=0}^{9} e^{-12} \cdot \frac{12^k}{k!}\right]\,.\end{aligned}$$

(c) We have

$$\begin{aligned}\mathbb{P}(N([0,1])=0\,|\,N([0,4])=13) &= \frac{\mathbb{P}(N([0,1])=0\,,\,N([0,4])=13)}{\mathbb{P}(N([0,4])=13)} = \\ &= \frac{\mathbb{P}(N([0,1])=0\,,\,N([1,4])=13)}{\mathbb{P}(N([0,4])=13)} = \\ &= \frac{\mathbb{P}(N([0,1])=0)\mathbb{P}(N([1,4])=13)}{\mathbb{P}(N([0,4])=13)} = \\ &= \frac{e^{-4}e^{-12}\frac{12^{13}}{13!}}{e^{-16}\frac{16^{13}}{13!}} = \left(\frac{3}{4}\right)^{13}.\end{aligned}$$

Ex. 20: Denote by $S_{10{,}000}$ the number of heads. Hence the number of tails is $10,000 - S_{10{,}000}$. So we want to compute

$$\mathbb{P}(|S_{10{,}000} - (10,000 - X)| \leq 100) = \mathbb{P}(|2S_{10{,}000} - 10,000| \leq 100)\,.$$

Recall that

$$|x| \leq a \Leftrightarrow -a \leq x \leq a\,.$$

So

$$\begin{aligned}&\mathbb{P}(|2S_{10{,}000} - 10,000| \leq 100) = \mathbb{P}(-100 \leq 2S_{10{,}000} - 10,000 \leq 100) = \\ &\quad = \mathbb{P}(9900 \leq 2S_{10{,}000} \leq 10,100) = \mathbb{P}(4950 \leq S_{10{,}000} \leq 5050) = \\ &\quad = \mathbb{P}(S_{10{,}000} \leq 5050) - \mathbb{P}(S_{10{,}000} < 4950)\,.\end{aligned}$$

Define for $i = 1, \ldots, 10,000$

$$X_i = \begin{cases} 1\,, & \text{if the } i\text{-th tossing is head} \\ 0\,, & \text{otherwise.}\end{cases}$$

Then $S_{10{,}000} = \sum_{i=1}^{10{,}000} X_i$. Since

$$\mathbb{E}[X_i] = 12\,, \qquad \mathrm{Var}(X_i) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\,,$$

using the CLT we get

$$\begin{aligned}&\mathbb{P}(S_{10{,}000} \leq 5050) - \mathbb{P}(S_{10{,}000} < 4950) = \\ &= \mathbb{P}\left(\frac{S_{10{,}000} - 5000}{\sqrt{2500}} \leq \frac{5050 - 5000}{\sqrt{2500}}\right) - \mathbb{P}\left(\frac{S_{10{,}000} - 5000}{\sqrt{2500}} \leq \frac{4950 - 5000}{\sqrt{2500}}\right) \approx \\ &\underset{CLT}{\approx} \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 2\cdot 0.84 - 1 = 0.08\,.\end{aligned}$$

Ex. 23: Define for $i = 1, \ldots, 10,000$

$$X_i = \begin{cases} 1\,, & \text{if the } i\text{-th battery has lifetime at most } 1100 \\ 0\,, & \text{otherwise.}\end{cases}$$

Then $S_{10,000} = \sum_{i=1}^{10,000} X_i$. Define $Y \sim \mathcal{N}(1.2 \cdot 10^3, 10^4)$. Since

$$\mathbb{P}(X_i = 1) = \mathbb{P}(Y \leq 1100) = \Phi\left(\frac{1100 - 1200}{100}\right) = \Phi(-1) = 1 - \Phi(1) \approx 0.84\,,$$

then

$$\mathbb{E}[X_i] = 0.84 \qquad \mathrm{Var}(X_i) = 0.84 \cdot (1 - 0.84) \approx 0.13\,.$$

Using the CLT we get

$$\mathbb{P}(S_{100} \geq 20) = \mathbb{P}\left(\frac{S_{100} - 100 \cdot 0.84}{\sqrt{100 \cdot 0.13}} \geq \frac{20 - 100 \cdot 0.84}{\sqrt{100 \cdot 0.13}}\right) \underset{CLT}{\approx} 1 - \Phi\left(-\frac{64}{\sqrt{13}}\right) =$$
$$= \Phi\left(\frac{64}{\sqrt{13}}\right) \approx 0\,.$$

Ex. 28: Let $X \sim \mathrm{Bin}(n,p)$. Let us approximate $\mathbb{P}(X < n)$ using the CLT. Note that $X = \sum_{i=1}^{n} Y_i$, where $Y_i \sim \mathrm{Ber}(p)$. Hence we get

$$\mathbb{P}(X < n) = \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq n\right) = \Phi\left(\frac{n - np}{\sqrt{np(1-p)}}\right) =$$
$$= \Phi\left(\sqrt{n} \cdot \frac{1-p}{\sqrt{p(1-p)}}\right) = \Phi\left(\sqrt{n} \cdot \sqrt{\frac{1-p}{p(1-p)}}\right)\,.$$

To compute the value of p that maximizes $\mathbb{P}(X = n)$ we need to compute the value of p that minimizes $\mathbb{P}(X < n)$. Since Φ and the square root are increasing functions, it is enough to compute the value of p that minimizes $\frac{1-p}{p}$. This happens for $p = 1$.

Ex. 32: Note that

$$\lambda^3 = \mathbb{E}[(X)_3] = \mathbb{E}[X(X-1)(X-2)] = \mathbb{E}[X(X^2 - 3X + 2)] = \mathbb{E}[X^3 - 3X^2 + 2X]\,.$$

Since $\mathbb{E}[X] = \lambda$ and $\mathbb{E}[X^2] = \mathrm{Var}(X) + \mathbb{E}[X]^2 = \lambda + \lambda^2$, we have

$$\lambda^3 = \mathbb{E}[X^3] - 3\mathbb{E}[X^2] + 2\mathbb{E}[X] \Rightarrow \mathbb{E}[X^3] = \lambda^3 + 3(\lambda + \lambda^2) - 2\lambda = \lambda^3 + 3\lambda^2 + \lambda\,.$$

Ex. 35: (a) Denote by $p = \frac{1}{2}$ the probability of heads. First observe that

$$\mathbb{P}(X = 0) = (1 - \mathbb{P}(10 \text{ heads}) - \mathbb{P}(10 \text{ tails})^{365}\,.$$

Indeed we are asking that for all 365 days we get a mixed configurations of heads and tails. We have

$$\mathbb{P}(X > 1) = 1 - \mathbb{P}(X = 0) = 1 - (1 - p^{10} - (1-p)^{10})^3 65 =$$
$$= 1 - (1 - p^{10} - (1-p)^{10})^3 65 = 1 - \left(1 - \frac{1}{2^9}\right)^{365} = 1 - \left(\frac{511}{512}\right)^{365} \approx 0.5\,.$$

(b) We approximate X as $Y\text{Poi}(\lambda)$ with $\lambda = 365 \cdot q$, where

$$q = p^{10} + (1-p)^{10} = \frac{1}{2^9}\,.$$

So $\lambda \approx 0.7$. Hence

$$\mathbb{P}(X > 1) \approx \mathbb{P}(Y > 1) = 1 - e^{-\lambda} = 1 - e^{-0.7} \approx 0.5\,.$$

Ex. 41: Define for $i = 1, \dots, 72$

$$X_i = \begin{cases} 1\,, & \text{if the } i\text{-th roll is a 6} \\ 0\,, & \text{otherwise.} \end{cases}$$

Define $S_{72} = \sum_{i=1}^{72} X_i$. We want to compute $\mathbb{P}(S_{72} = 3)$. Using the Poisson approximation we have

$$S_{72} \approx Y \sim \text{Poi}\left(72 \cdot \frac{1}{6}\right) = \text{Poi}(12)\,.$$

Hence

$$\mathbb{P}(Y = 3) = e^{-12}\frac{12^3}{3!} = 288e^{-12} \approx 0.0018\,.$$

With the normal approximation instead we have to notice $\mathbb{E}[X] = \frac{1}{6}$ and $\text{Var}(X) = \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$. Hence using the continuity correction we get

$$\mathbb{P}(S_{72} = 3) = \mathbb{P}(2.5 \le S_{72} \le 3.5) \underset{CLT}{\approx} \Phi\left(\frac{3.5 - 12}{\sqrt{10}}\right) - \Phi\left(\frac{2.5 - 12}{\sqrt{10}}\right) \approx$$
$$\approx \Phi(-2.69) - \Phi(-3) = \Phi(3) - \Phi(2.69) \approx 0.9987 - 0.9964 = 0.0023.$$

Ex. 44: Define for $i = 1, \dots, 400$

$$X_i = \begin{cases} 1\,, & \text{if the } i\text{-th watch is defective} \\ 0\,, & \text{otherwise.} \end{cases}$$

Define $S_{400} = \sum_{i=1}^{400} X_i$. Then S_{400} denotes the number of defective watches and $S_{400} \sim \text{Bin}(400, 1/2)$.

(a) We have

$$\mathbb{P}(S_{400} \ge 215) = \sum_{k=215}^{400} \binom{400}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{400-k} = \sum_{k=215}^{400} \binom{400}{k} \left(\frac{1}{2}\right)^{400}\,.$$

(b) Note that $\mathbb{E}[X_1] = \frac{1}{2}$ and $\text{Var}(X_1) = \frac{1}{4}$. Using the CLT we get

$$\mathbb{P}(S_{400} \ge 215) = \mathbb{P}\left(\frac{S_{400} - 400 \cdot \frac{1}{2}}{\sqrt{400 \cdot \frac{1}{4}}} \ge \frac{215 - 400 \cdot \frac{1}{2}}{\sqrt{400 \cdot \frac{1}{4}}}\right) \approx$$
$$\underset{CLT}{\approx} 1 - \Phi(1.5) \approx 1 - 0.93 = 0.07\,.$$