$\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 = \sum_{i=1}^{n} \sum_{j=1}^{i} (j-1+1) = \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{n(n+1)(n+2)}{6}$

Geometric $Geo(p)$ $X \in \{1, 2, 3 \ldots\}$ time of the first head

$P(X=k) = (1-p)^{k-1} p$

$P(X>k) = \sum_{j=k+1}^{\infty} (1-p)^{j-1} p = p \frac{(1-p)^k}{1-(1-p)} = (1-p)^k$

$P(X>k)P(X>j) = (1-p)^k (1-p)^j = (1-p)^{k+j}$

$= P(X > k+j)$

$P(X>j) = \frac{P(X>k+j)}{P(X>k)} = \frac{P(X>k+j, X>k)}{P(X>k)}$

$= P(X>k+j \mid X>k)$

$P_{XY}(a,b) = \begin{cases} \frac{1}{ab} & 1 \le b \le a \le 4 \\ 0 & \text{otherwise} \end{cases}$ $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \approx \log n$

$P(Y=b) = \sum_{a \in \mathbb{R}} P_{XY}(a,b) = \sum_{a=b}^{4} \frac{1}{ab} = \frac{1}{b} \sum_{a=b}^{4} \frac{1}{a} = \frac{1}{b}\left(\frac{1}{b} + \frac{1}{b+1} + \cdots + \frac{1}{4}\right)$

$= \frac{h_4 - h_{b-1}}{4}$

Poisson
Binomial
Geometric
Exponential
Normal

Bernoulli
Uniform
Gamma

Gamma
Exponential
$f_X(x) = \lambda e^{-\lambda x}$

$k = nx$
$\frac{1}{n} = \lambda$

$\frac{1}{n} \sum_{i=1}^{n} e^{-\lambda x_i} \to E e^{-\lambda X}$ $(1-E)^n$

$X \sim Geo(p)$
$f(x, p) = (1-p)^{x-1} p$
exponential

Gamma
Exponential $\lambda e^{-\lambda x}$ density
$\sum_{i=1}^{n} \log(1-F(x_i)) = \sum_{i=1}^{n} \log(1-(1-e^{-\lambda x_i}))$
$\sum_{i=1}^{n} \log(1-e^{-\lambda x_i}) = \sum_{i=1}^{n} \log e^{-\lambda x_i}$

$\Gamma(1) = 1$ $\Gamma(n) = (n-1)!$ $\Gamma(n+1) = n!$ $\Gamma(x)$

$Y, \lambda X$
Y has density $\frac{\lambda^{\alpha} e^{-\lambda y}}{\Gamma(\alpha)} (\lambda y)^{\alpha-1}$
$x = 1$ $\lambda e^{-\lambda y}$ (Exp) Exponential
$e^{-y}, y^{\alpha-1}$ $\frac{(\lambda)^{\alpha} y^{\alpha-1} e^{-\lambda y}}{\Gamma(\alpha)}$ χ^2 density

$x = \Gamma(x)$
$\int_0^{\infty} e^{-x} x^{\alpha-1} dx = \Gamma(\alpha)$
$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$

Exponential $X \sim Exp(\lambda)$
$M_X(t) = \frac{\lambda}{\lambda - t}$ $X \sim Exp(\lambda)$
$Y \sim Exp(\lambda)$

$X \perp\!\!\!\perp Y$
$M_{X+Y}(t) = M_X(t) M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^2$ $Y \sim \Gamma(\alpha, \lambda)$
$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$ $M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\beta}$

For which $X \sim \Gamma(\alpha, \lambda)$ with $\Gamma(\alpha, \lambda) \perp\!\!\!\perp \Gamma(\beta, \lambda)$ $M_X(t) M_Y(t) = M_Z(t)$ is exponential
Poisson
Gamma
Bernoulli

X_1, X_2, X_3 iid $Ber(\lambda)$, $\Gamma(1, \lambda)$
$\Gamma_1 \sim Y_1 + \cdots + Y_n \sim \Gamma(n, \lambda)$ Erlang distribution

$X \sim Ber(p)$ $P(X=1) = p$, $P(X=0) = 1-p$
$E[e^{tX}] = e^t P(X=1) + e^0 P(X=0)$
$= 1 - p + pe^t$

$X \sim Bin(n, p)$
$X = X_1 + \cdots + X_n$ $X_1, \ldots, X_n$ iid $Ber(p)$
$E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$
$X \sim B(n, p)$ $\perp\!\!\!\perp$ $Y \sim B(m, p)$
$X+Y$ has MGF $(1-p+pe^t)^n \cdot (1-p+pe^t)^m = (1-p+pe^t)^{n+m}$ $B(n+m, p)$

Geometric $X \in \{1, 2, 3 \ldots\}$
$X \sim Geo(p)$
$P(X=k) = (1-p)^{k-1} p$

$M_X(t) = E[e^{tX}] = \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} p$
$= p \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} = \frac{p}{1-p} \sum_{k=1}^{\infty} e^{tk} (1-p)^k$
$= \frac{p}{1-p} \sum_{k=1}^{\infty} (e^t (1-p))^k = \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1 - e^t(1-p)} = \frac{pe^t}{1 - e^t(1-p)}$ at 0 $\frac{p}{1-(1-p)} = 1$

$e^t(1-p) < 1$
$e^t < \frac{1}{1-p}$
$t < \log\left(\frac{1}{1-p}\right)$

$M_U(t) = \int_0^1 e^{tu} du = \frac{e^{tu}}{t}\Big|_0^1 = \frac{e^t - 1}{t} = \frac{1 + t + \frac{1}{2}t^2 + \cdots - 1}{t}$

$e^t = 1 + t + \frac{1}{2}t^2 + \cdots$
$e^t - 1 = t + \frac{1}{2}t^2 + \cdots$

$\left(\frac{e^t - 1}{t}\right)^n$

one mode

unimodal $X \perp\!\!\!\perp Y$ $U \perp\!\!\!\perp Y$

7.15 $X \sim U(\{1, \ldots, n\})$ $n = 6$
$Y \sim U(\{1, \ldots, m\})$ $m = 6$ two die

$X \perp\!\!\!\perp Y$ Find pmf $X+Y$

$X+Y = \{2, 3, \ldots, n+m\}$

$m \le n$

nm squares
each with prob $\frac{1}{nm}$

$2 \le k \le m$ $X+Y = 2$ $P(X+Y=2) = \frac{1}{nm}$
$P(X+Y=k) = \frac{k-1}{nm}$ $P(X+Y=3) = \frac{2}{nm}$

$P(X+Y=k) = \frac{k-1}{nm}$, $2 \le k \le m$

$P(X+Y=k) = \frac{m}{nm} = \frac{1}{n}$, $m+1 \le k \le n+1$

$P(X+Y=k) = \frac{n+m-k+1}{nm}$, $m+2 \le k \le n+m$

$P(X+Y=m) = \frac{m-1}{nm}$
$P(X+Y=m+1) = \frac{m}{nm}$
$P(X+Y=m+2) = \frac{m}{nm}$
$\vdots$
$P(X+Y=n+1) = \frac{m}{nm}$

$P(X+Y=n+2) = \frac{m-1}{nm}$
$P(X+Y=n+3) = \frac{m-2}{nm}$
$\vdots$
$P(X+Y=n+m) = \frac{1}{nm}$

$\frac{1}{2}$
$\frac{2}{3}$
$\frac{3}{4}$
$\frac{4}{5}$

$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$

# of diagonal $X+Y=3$ 2 way
m $X+Y=4$ 2 ways
3rd $X+Y=5$ 1 ways

$X+Y=n+1$
m ways

Poisson $e^{\lambda(e^t - 1)}$
Normal $e^{\frac{1}{2}\sigma^2 t^2 + \mu t}$

9.2 Let $X \sim Exp(\lambda)$ $E[X] = \frac{1}{\lambda}$, $t > 0$
a) $P(X > t) \le \frac{E[X]}{t}$ $P(X > t) \le \frac{E[X]}{t}$
$= \frac{2}{4} = \frac{1}{2}$

7.4 $X \perp\!\!\!\perp Y$
$X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

density $X+Y$
density X $\lambda e^{-\lambda x} 1_{(x>0)}$
density Y $\mu e^{-\mu x} 1_{(x>0)}$

density of $X+Y$

$f_{X+Y}(v) = \int_{-\infty}^{\infty} f_{X,Y}(x, v-x) dx$ $P(X+Y=v) = \sum_{x \in \mathbb{R}} P(X=x, Y=v-x)$

under $\perp\!\!\!\perp$
$= \int_{-\infty}^{\infty} f_X(x) f_Y(v-x) dx$
$= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} 1_{(x>0)} \mu e^{-\mu(v-x)} 1_{(v-x>0)} dx$

$1_{(x>0)}$
$1_{(v-x>0)}$
$v > x$
$0 < x < v$

$= \mu\lambda \int_0^v e^{-x(\mu-\lambda) - \mu v} dx$ exponent $-\lambda x - \mu(v-x)$
$= -\lambda x - \mu v + \mu x$
$= x(\mu - \lambda) - \mu v$

$= \mu\lambda e^{-\mu v} \int_0^v e^{x(\mu-\lambda)} dx$

$= \mu\lambda e^{-\mu v} \frac{e^{x(\mu-\lambda)}}{\mu - \lambda}\Big|_0^v$ $\mu > \lambda$
$\mu < \lambda$

$= \frac{\mu\lambda e^{-\mu v}}{\mu - \lambda}\left(e^{v(\mu-\lambda)} - 1\right)$

$= \frac{\mu\lambda}{\mu - \lambda}\left(e^{-\lambda v} - e^{-\mu v}\right)$ $v \ge 0$

$\lambda \ne \mu$ $x, v \ge 0$