$P(X_1=k_1, X_2=k_2)$

$\text{union} \perp\!\!\!\perp = P(X_1=k_1, X_2=k_2, X_3=\dots=X_n=k_n)$

$\binom{n}{k_1} p_1^{k_1}(1-p_1)^{n-k_1} \times \binom{n}{k_2} p_2^{k_2}(1-p_2)^{n-k_2} = \binom{n}{k_1,\, n-k_1-k_2} p_1^{k_1} p_2^{k_2} (1-p_1-p_2)^{n-k_1-k_2}$

$0 = P(X_1=k_1, X_2=k_2)$

$= P(X_1=n, X_2=1)$ not independent $p_1+p_2 \neq 1$, $0<p_1,p_2<1$

5.31 $U \sim U[0,1]$ $Y = e^{\frac{U}{1-U}}$

Find density of Y

$t>1$ since $F_Y(t)=0$ $t<1$, $t \geq 1$

$F_Y(t) = P(Y \leq t) = P(e^{\frac{U}{1-U}} \leq t)$

$= P\left(\frac{U}{1-U} \leq \log t\right) = P(U \leq (1-U)\log t)$ $0 \leq U \leq 1$

$= P(U \leq \log t - U\log t)$ $0 \leq \frac{U}{1-U} < \infty$

$= P(U(1+\log t) \leq \log t)$ $1 = e^0 = e^{\frac{U}{1-U}} < \infty$, $Y \geq 1$, $t \in [1,\infty)$

$= P\left(U \leq \frac{\log t}{1+\log t}\right) = \frac{\log t}{1+\log t}$ $t>1$, $\log t > 0$

$0 < \frac{\log t}{1+\log t} < 1$

Reality check

$F(1) = \frac{\log 1}{\log(1+1)} = 0$, $t \to \infty$

Density $\frac{d}{dt}\left(\frac{\log t}{1+\log t}\right)$ in $[1,\infty)$ and 0 otherwise

$\frac{X-a}{b-a}$ on $[0,1]$

$X = (b-a)U + a$

* optional

$X \perp\!\!\!\perp Y$ $X \sim Poi(\lambda)$ $Y \sim Poi(\mu)$

$X, Y \in \{0,1,2,\dots\}$

$X+Y \in \{0,1,2,\dots\}$ convolution

$P(X+Y=k) = \sum_{j=0}^{k} P(X=j, Y=k-j)$

$= \sum_{j=0}^{k} P(X=j)P(Y=k-j) = \sum_{j=0}^{k} e^{-\lambda}\frac{\lambda^j}{j!} e^{-\mu}\frac{\mu^{k-j}}{(k-j)!}$

$= \frac{e^{-(\lambda+\mu)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \mu^{k-j} = \frac{e^{-(\lambda+\mu)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \mu^{k-j}$

$= \frac{e^{-(\lambda+\mu)}}{k!} (\lambda+\mu)^k$

$X+Y \sim P(\lambda+\mu)$

MGF $M_X(t) = e^{\lambda(e^t-1)}$ $M_Y(t) = e^{\mu(e^t-1)}$

$M_{X+Y}(t) \overset{?}{=} M_X(t)M_Y(t) = e^{(\lambda+\mu)(e^t-1)}$

$W(X) = 1(X \leq t)$ $g(Y) = 1(Y \leq t)$

$X \perp\!\!\!\perp Y \iff \forall h,g \quad \mathbb{E}[h(X)g(Y)] = \mathbb{E}[h(X)]\mathbb{E}[g(Y)]$

$F_{X,Y}(s,t) = F_X(s)F_Y(t) \iff \forall h,g \quad h(X) \perp\!\!\!\perp g(X)$

$P_{X,Y}(x,y) = P_X(x)P_Y(y)$

$\mathbb{E}[g(X)h(Y)] = \sum_{a \in \mathcal{A}, b \in \mathcal{B}} g(a)h(b)P_X(a)P_Y(b) \overset{\perp\!\!\!\perp}{=} \sum_{a \in \mathcal{A}, b \in \mathcal{B}} g(a)h(b)P_X(a)P_Y(b)$

$= \left(\sum_{a \in \mathcal{A}} g(a)P_X(a)\right)\left(\sum_{b \in \mathcal{B}} h(b)P_Y(b)\right) = \mathbb{E}\,g(X)\,\mathbb{E}\,h(Y)$

$X \perp\!\!\!\perp Y \Rightarrow e^{tX} \quad e^{tY}$

$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX}e^{tY}] = \mathbb{E}[e^{tX}]\mathbb{E}[e^{tY}] = M_X(t)M_Y(t)$

$X \perp\!\!\!\perp Y \Rightarrow M_{X+Y}(t) = M_X(t)M_Y(t)$

$X \perp\!\!\!\perp Y$

$\Gamma(\lambda,\alpha)$ $\Gamma(\mu,\beta)$ $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}$ $\Gamma(\lambda,\alpha)$

$X+Y \sim \Gamma(\lambda,\alpha+\beta)$ $M_Y(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\beta}$ $\Gamma(\lambda,\beta)$

$M_{X+Y}(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha+\beta}$ $\Gamma(\lambda,\alpha+\beta)$

Normal $X \sim N(\mu,\sigma^2)$ $M_X(t) = e^{\frac{1}{2}\sigma^2t^2+\mu t}$

$Y \sim N(\tau,\gamma^2)$ $M_Y(t) = e^{\frac{1}{2}\gamma^2t^2+\tau t}$

Previously $M_{X+Y}(t) = e^{\frac{1}{2}(\sigma^2+\gamma^2)t^2+(\mu+\tau)t}$

Therefore $X+Y \sim N(\mu+\tau, \sigma^2+\gamma^2)$

Convolution

Density $f_{X+Y}(v) = \int_{-\infty}^{\infty} f_{X,Y}(x, v-x)\,dx$

$X \perp\!\!\!\perp Y$ $= \int_{-\infty}^{\infty} f_X(x)f_Y(v-x)\,dx$

$v = f_X, f_Y(x), f_Y(s)$

Normal $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} e^{-\frac{1}{2\gamma^2}(v-x-\tau)^2} dx$

Sums of independent Poissons are Poisson

" " Normals are Normals

" " Gammas (with the same scale parameter) are Gamma

Sums of uniforms is not uniform.

$N(0,1)$ Convolution

MGF not to know!)

Find: Y, Y_1, Y_2 are random variables with MGF $M_Y(t), M_{Y_1}(t), \dots$

such that $\lim_{n\to\infty} M_{Y_n}(t) = M_Y(t)$ for some Y

then Y_n converges in distribution to Y

If Y is continuous then $\lim_{n\to\infty} P(Y_n \leq t) = P(Y \leq t)$

X_1, X_2, X_3 iid

$S_n = X_1+X_2+\dots+X_n$ $M_{S_n}(t) = M_X(t)M_X(t)$

$M_{S_n}(t) = \mathbb{E}[e^{t(X_1+\dots+X_n)}] = \mathbb{E}[e^{tX_1}\cdots e^{tX_n}] \overset{\perp\!\!\!\perp}{=} M_X(t)\cdots M_X(t) = M_X(t)^n$

$M_{S_n}(t) = (M_X(t))^n$

Generic X has mean 0 variance 1 $V_n = \frac{S_n}{\sqrt{n}}$ mean 0 variance 1

$\frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\sum(X_i-\mu)/\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n}}V_n$

So S_n is iid mean 0 variance 1 $\frac{S_n}{\sqrt{n}}$ converges to $N(0,1)$

$M_{S_n/\sqrt{n}}(t) = \mathbb{E}\left[e^{t(S_n/\sqrt{n})}\right] = M_{S_n}\left(\frac{t}{\sqrt{n}}\right) = \left(M_X\left(\frac{t}{\sqrt{n}}\right)\right)^n$

$M_X(t) = 1 + \cancel{tM_X'(0)}^{\,0} + \frac{1}{2}t^2 + o(t^2) = 1 + \frac{1}{2}t^2 + o(t^2)$

$M_X(0) = 1$

$M_X'(0) = \mu$

$M_X''(0) = 1$

$\left(1+\frac{1}{2n}t^2+o\left(\frac{t^2}{n}\right)\right)^n$

$\left(1+\frac{1}{2n}t^2\right)^n \longrightarrow e^{t^2/2}$

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\dots$ $\left(1+\frac{x}{n}\right)^n \to e^x$

$e^{tX} = \sum_{n=0}^{\infty}\frac{(tX)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^nX^n}{n!} = 1+tX+\frac{1}{2}t^2X^2+\dots$

$\mathbb{E}[e^{tX}] = \sum_{n=0}^{\infty}\frac{t^n\mathbb{E}[X^n]}{n!} = 1+\cancel{t\mathbb{E}[X]}^{\,0}+\frac{1}{2}t^2\mathbb{E}[X^2]+\frac{1}{6}t^3\mathbb{E}[X^3]+\dots$

$M_X(t) = \mathbb{E}[e^{tX}] = e^{\frac{1}{2}t^2}$ $1+\frac{1}{2}t^2 \to 1+\frac{t^2}{2n}$

$\left(1+\frac{t^2}{2n}\right)^n \to e^{t^2/2}$

$\varphi_X(t) = \mathbb{E}[e^{itX}] = e^{\frac{1}{2}t^2}$ exists for all X

Characteristic function

$|e^{itX}| = 1$