Joint Distribution

$X=(X_1,X_2,\dots,X_n)\in\mathbb{R}^n$

Domain

Joint $e=3$

$P(X_1=a_1,X_2=a_2,\dots,X_n=a_n)=P(\{a_1,a_2,\dots,a_n\})$

$P(X=a)$

$E[g(X,X_2)]=\sum_{a_1\in\mathcal{A}_1}\sum_{a_2\in\mathcal{A}_2} g(a_1,a_2)\,P_{X_1X_2}(a_1,a_2)$

$E[g(X)]=\sum_{a\in\mathcal{A}} g(a)P(a)$

$P(X=a_1)=\sum_{(a_2,a_3,\dots,a_n)\in\mathcal{A}_2\times\dots\times\mathcal{A}_n}P(X_1=a_1,X_2=a_2,\dots,X_n=a_n)$

Marginal

$P(X_1=a_1,X_2=a_2)=\sum_{(a_3,\dots,a_n)\in\mathcal{A}_3\times\dots\times\mathcal{A}_n}P(X_1=a_1,X_2=a_2,X_3=a_3,\dots,X_n=a_n)$

Bivariate not required

$P(A)=\sum_{j=1}^{m}P(A\cap B_j)$

Binomial $X=X_1+\dots+X_n$ $\quad Bin(m,p)$ $\quad X_i\in\{0,1\}$

Throw m balls into bins

$r=2$

$X_i=$ # balls in bin i

$P(X=a)=\binom{m}{a_1,\dots,a_r}p_1^{a_1}\cdots p_r^{a_r}$

$\binom{n}{a_1,\dots,a_r}=\frac{n!}{a_1!\cdots a_r!}$

n ways choose a_1, $(n-a_1)$ ways choose a_2, ...

$X_1\sim Bin(m,p_1)$

$(X_1,X_2)\sim$ multinomial

density $f(x)$ random variable

(X,Y) joint density function $f(x,y)\geq 0$

$\iint_{-\infty}^{\infty}f(x,y)\,dx\,dy=1$

$B=(-\infty,t)$

CDF

$P(X\in B)=\int_B f(u)\,du$

$\boxed{P((X,Y)\in B)}$

$B\subseteq\mathbb{R}^2$

$=\iint_B f(x,y)\,dx\,dy$

$B=(-\infty,1)\times(-\infty,5]$

$f(x,y)=\begin{cases}\frac{3}{2}(xy^2+y), & 0\leq x\leq 1,\ 0\leq y\leq 1\\ 0 & \text{otherwise}\end{cases}$

nonnegative

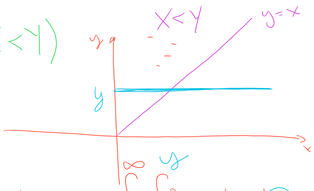
$\frac{3}{2}\int_0^1\int_0^1(xy^2+y)\,dx\,dy=\frac{3}{2}\int_0^1\left(\frac{1}{2}x^2y^2+yx\right)\Big|_{x=0}^{x=1}dy$

$=\frac{3}{2}\int_0^1\left(\frac{1}{2}y^2+y\right)dy=\frac{3}{2}\left(\frac{y^3}{6}+\frac{1}{2}y^2\right)\Big|_0^1=\frac{3}{2}\left(\frac{1}{6}+\frac{1}{2}\right)=1$

$f_X(x)=\int_{-\infty}^{\infty}f(x,y)\,dy=\frac{3}{2}\int_0^1(xy^2+y)\,dy=\frac{3}{2}\left(\frac{xy^3}{3}+\frac{1}{2}y^2\right)\Big|_{y=0}^{y=1}=\frac{3}{2}\left(\frac{x}{3}+\frac{1}{2}\right)\quad 0\leq x\leq 1$

$P(X<Y)$

$X\geq 0,\ Y\geq 0$



$X=Y$

$P(X<Y)=\iint_B f_{X,Y}(x,y)\,dx\,dy=\int_0^{\infty}\int_0^{y}f(x,y)\,dx\,dy$

Our example

$\frac{3}{2}\int_0^1\int_0^y(xy^2+y)\,dx\,dy$

Special case: $X_1,X_2,\dots,X_n$ independent

Defn $P(X_1\in B_1,X_2\in B_2,\dots,X_n\in B_n)=P(X_1\in B_1)\cdots P(X_n\in B_n)$

A and B are $\perp\!\!\!\perp$

when $P(A\cap B)=P(A)P(B)$

$F_X(t_1,t_2,\dots,t_n)=P(X_1\leq t_1,\dots,X_n\leq t_n)=\prod_{j=1}^{n}P(X_j\leq t_j)=\prod_{j=1}^{n}F_{X_j}(t_j)$

$\Updownarrow$

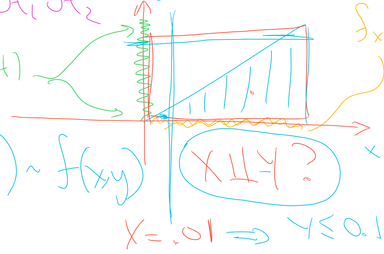
$f_X(t)=\prod_{j=1}^{n}f_{X_j}(t_j)=\frac{\partial^n F_X(t_1,\dots,t_n)}{\partial t_1\cdots\partial t_n}$

$F_{X,Y}(t_1,t_2)=\int_{-\infty}^{t_2}\left[\int_{-\infty}^{t_1}f_{X,Y}(x,y)\,dx\,dy\right]$

$\frac{\partial F_{X,Y}(t_1,t_2)}{\partial t_2}=\int_{-\infty}^{t_1}f_{X,Y}(x,t_2)\,dx$

$\frac{\partial^2 F_{X,Y}(t_1,t_2)}{\partial t_1\partial t_2}=f_{X,Y}(t_1,t_2)$

$f_X(t)\,f_Y(s)\neq f_{X,Y}(t,s)$



$f_Y(t)$

$f_X(t)$

$(X,Y)\sim f(x,y)$

$X\perp\!\!\!\perp Y$?

$f(x,y)$ is positive on this triangle, 0 otherwise

$X=.01\Rightarrow Y\leq 0.1$