EX. 69 (EX. SHEET 3)

$Z \sim N(0,1)$ $\qquad$ m positive integer

$\mathbb{E}[Z^m]$ = m-th moment of Z

$$\mathbb{E}[Z^m] = \int_{-\infty}^{+\infty} z^m \cdot f(z)\,dz$$

where $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$$\mathbb{E}[Z^m] = \int_{-\infty}^{+\infty} z^m \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{z^{m-1}}_{g(x)} \cdot \underbrace{z e^{-\frac{z^2}{2}}}_{f'(x)}\,dz \stackrel{\text{Integ. by parts}}{=}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\underbrace{\left[z^{m-1} \cdot \left(-e^{-\frac{z^2}{2}}\right) \right]_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{+\infty} (m-1) z^{m-2} \left(-e^{-\frac{z^2}{2}}\right) dz \right)$$

$$= (m-1) \int_{-\infty}^{+\infty} z^{m-2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = (m-1)\,\mathbb{E}[Z^{m-2}]$$

So:

$$\mathbb{E}[Z^m] = (m-1)\,\mathbb{E}[Z^{m-2}]$$

$$= (m-1)(m-3)\,\mathbb{E}[Z^{m-4}]$$

$$= (m-1)(m-3)(m-5)\,\mathbb{E}[Z^{m-6}]$$

$$\mathbb{E}[Z^m] = \begin{cases} (m-1)(m-3)(m-5)\ldots\ldots \mathbb{E}[Z^0] & \text{if } m \text{ even} \quad (\mathbb{E}[Z^0] = 1) \\ (m-1)(m-3)(m-5)\ldots\ldots \mathbb{E}[Z^1] & \text{if } n \text{ odd} \quad (\mathbb{E}[Z^1] = 0) \end{cases}$$

$$\mathbb{E}[Z^m] = \begin{cases} (m-1)!! & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

$$\mathbb{E}[Z^2] = 1!! = 1$$

$$\mathbb{E}[Z^6] = 5 \cdot 3 \cdot 1 = 15$$

Ex.

X cont. r.v. with prob. density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \in [0,1) \\ 2-x & \text{if } x \in [1,2) \\ 0 & \text{if } x \geq 2 \end{cases}$$

$$\mathbb{P}(X < 3/2) = \int_{-\infty}^{3/2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^1 x\,dx + \int_1^{3/2} (2-x)\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^{3/2}$$

$$= \frac{1}{2} + 3 - \frac{9}{8} - \left(2 - \frac{1}{2}\right)$$

$$= \frac{7}{8}$$

CDF = ?

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f(x)\,dx$$

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1) \\ 2-x & x \in [1,2) \\ 0 & x \geq 2 \end{cases}$$

IF $t < 0$: $F_X(t) = \int_{-\infty}^{t} \underbrace{f(x)}_{0}\,dx = 0$

If $t \in [0,1)$: $F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \int_0^t \underbrace{f(x)}_{x}\,dx + \int_{-\infty}^{0} \underbrace{f(x)}_{0}\,dx$

$$= \frac{x^2}{2}\Big|_0^t = \frac{t^2}{2}$$

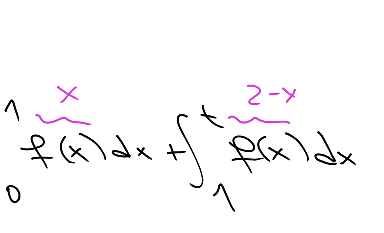
If $t \in [1,2)$

$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{0} \underbrace{f(x)}_{0}\,dx + \int_0^1 \underbrace{f(x)}_{x}\,dx + \int_1^t \underbrace{f(x)}_{2-x}\,dx$$

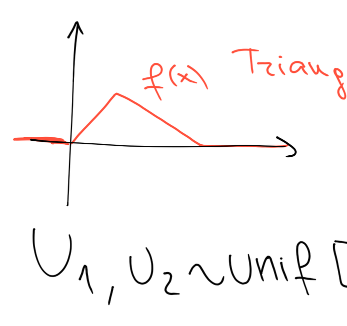
$$= \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^t$$

$$= \frac{1}{2} + 2t - \frac{t^2}{2} - \left(2 - \frac{1}{2}\right) = 2t - \frac{t^2}{2} - 1$$

if $t \geq 2$: $F_X(t) = \int_{-\infty}^{t} f(x)\,dx = 1$



$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t^2}{2} & \text{if } t \in [0,1) \\ -\frac{t^2}{2} + 2t - 1 & \text{if } t \in [1,2) \\ 1 & \text{if } t \geq 2 \end{cases}$$



Triangular density function.

$U_1, U_2 \sim \text{Unif}[0,1]$ indep.

$Z = U_1 + U_2$

Z has density $f(x)$

Ex. 58 (EXERCISE SHEET 3)

$X \sim \text{Bin}(m,p)$ $\qquad$ $\mathbb{E}\left[\frac{1}{X+1}\right]$

$$\mathbb{E}[X] = \sum_{k=0}^{m} \frac{1}{k+1} \cdot \mathbb{P}(X=k) =$$

$$= \sum_{k=0}^{m} \frac{1}{k+1} \cdot \binom{m}{k} p^k (1-p)^{m-k} \qquad \binom{m}{k} = \frac{m!}{k!(m-k)!}$$

$$= \sum_{k=0}^{m} \frac{1}{k+1} \cdot \frac{m!}{k! \cdot (m-k)!} p^k (1-p)^{m-k}$$

$$= \sum_{k=0}^{m} \frac{m!}{(k+1)!(m-k)!} p^k (1-p)^{m-k}$$

$$= \sum_{k=0}^{m} \frac{1}{m+1} \cdot \underbrace{\frac{(m+1)!}{(k+1)!(m-k)!}}_{\binom{m+1}{k+1}} p^k (1-p)^{m-k}$$

$$= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k+1} p^k (1-p)^{m-k}$$

$$\sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} = \sum_{k=0}^{m} \mathbb{P}(X=k) = 1 \quad (X \sim \text{Bin}(n,p))$$

$X \sim \text{Bin}(n+1,p)$

$$= \frac{1}{m+1} \cdot \frac{1}{p} \sum_{k=0}^{m} \binom{m+1}{k+1} p^{k+1} (1-p)^{m-k} \stackrel{j=k+1}{=} \quad k = j-1$$

$$= \frac{1}{m+1} \cdot \frac{1}{p} \sum_{j=1}^{m+1} \underbrace{\binom{m+1}{j} p^j (1-p)^{m+1-j}}_{\mathbb{P}(\text{Bin}(n+1,p)=j)}$$

$$= \frac{1}{(m+1)p} \sum_{j=1}^{m+1} \mathbb{P}(\text{Bin}(n+1,p)=j) =$$

$$= \frac{1}{(m+1)p} \left[\underbrace{\sum_{j=0}^{m+1} \mathbb{P}(\text{Bin}(n+1,p)=j)}_{1} - \underbrace{\mathbb{P}(\text{Bin}(n+1,p)=0)}_{(1-p)^{m+1}} \right]$$

$$= \frac{1}{(m+1)p} \left(1 - (1-p)^{n+1}\right)$$

$$\mathbb{E}\left[\frac{1}{1+X}\right] \neq \frac{1}{1+\mathbb{E}[X]}$$

EX. 15

$\mathbb{E}[X] = 3 \qquad Var(X) = 4$

$\mathbb{E}[X^2] = ?$ $\qquad$ $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$= 4 + 9 = 13$ $\qquad$ $\Rightarrow \mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2$

$$Var(4X+2) = 4^2 Var(X) = 64$$

$$Var(aX+b) = a^2 Var(X).$$

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if X is a non-negative r.v.

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X^r]}{a^r} \quad r > 0 \qquad \text{Markov inequality}$$

Example

$X \sim \text{Bin}(100, 1/2)$

$$\mathbb{P}(X \geq 60) \stackrel{\text{Markov } (r=1)}{\leq} \frac{100 \cdot 1/2}{60} = 5/6 \approx 0.83$$

If $r = 2$?

$$\mathbb{P}(X \geq 60) \leq \frac{\mathbb{E}[X^2]}{60^2} = \frac{25 + 50^2}{60^2} = \frac{2525}{3600} \approx 0.7$$

$X \sim \text{Bin}(n,p)$

$\mathbb{E}[X] = mp$

$Var(X) = mp(1-p)$

$Var(X) = 100 \cdot \frac{1}{2} \cdot \frac{1}{2} = 25$

$\mathbb{E}[X] = 100 \cdot \frac{1}{2} = 50$

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X^r]}{a^r} \qquad \text{Markov with } r \text{ moment}$$

$X = |Y - \mathbb{E}[Y]|$

Apply Markov with moment 2 for $X = |Y - \mathbb{E}[Y]|$

$$\mathbb{P}(|Y - \mathbb{E}[Y]| \geq a) \leq \frac{\mathbb{E}\left[|Y - \mathbb{E}[Y]|^2\right]}{a^2} = \frac{Var(Y)}{a^2}$$

Chebyshev inequality:

$$\mathbb{P}(|Y - \mathbb{E}[Y]| \geq a) \leq \frac{Var(Y)}{a^2}$$