Ex. 1

1)
$$\mathbb{P}(X_A + X_B = 10) = \frac{\#\{(8,2),(8,2),(6,4)\}}{36} =$$
$$= \frac{3}{36} = \frac{1}{12}$$

2)
$$\mathbb{P}(X_A + X_B = 8) = \frac{\#\{(4,4),(5,3),(5,3),(6,2),(6,2)\}}{36}$$
$$= \frac{5}{36}$$

3)
$$\mathbb{P}(X_A > X_B) = \frac{\#\{(2,1),(2,1),(3,1),(3,1),(4,1),(4,3)\}}{36}$$
$$= \frac{6}{36} = \frac{1}{6}.$$

Ex. 2

1) $\mathbb{P}(\text{Italia fails}) = \mathbb{P}(3 \text{ failures}) + \mathbb{P}(4 \text{ failures})$
$$= 4p^3(1-p) + p^4$$

2) $\mathbb{P}(\text{America fails}) = \mathbb{P}(2 \text{ failures}) = p^2$

3)
$$4p^3(1-p) + p^4 < p^2 \Rightarrow 4p(1-p) + p^2 < 1 \Rightarrow$$
$$\Rightarrow 4p - 4p^2 + p^2 - 1 < 0 \Rightarrow 3p^2 - 4p + 1 > 0$$
$$\Rightarrow p < \frac{1}{3} \text{ or } p > 1.$$ Since p is a probability $\Rightarrow p \in (0, \frac{1}{3})$.

Ex. 3

$A = \{2B, 4R\}$ $\quad$ $B = \{5B, 3R\}$

1) $\mathbb{P}(1R, 1B \text{ from } A) = \dfrac{\binom{2}{1}\cdot\binom{4}{1}\cdot 2!}{6\cdot 5} = \dfrac{8}{15}$

2) $\mathbb{P}(1R, 1B \text{ from } B) = \dfrac{\binom{5}{1}\binom{3}{1}\cdot 2!}{8\cdot 7} = \dfrac{15}{28}$

3) $E = \{1R, 1B\}$ $\quad$ $A = \{\text{urn } A \text{ is chosen}\}$

$$\mathbb{P}(A|E) = \frac{\mathbb{P}(E|A)\,\mathbb{P}(A)}{\mathbb{P}(E|A)\,\mathbb{P}(A) + \mathbb{P}(E|A^c)\,\mathbb{P}(A^c)} =$$

$$= \frac{8/15 \cdot 1/2}{\frac{8}{15}\cdot\frac{1}{2} + \frac{15}{28}\cdot\frac{1}{2}} = \frac{\frac{8}{15}}{\frac{224+225}{420}} =$$

$$= \frac{8}{15}\cdot\frac{420}{449} = \frac{224}{449} \approx 0.5$$

Ex. 4

1) $\mathbb{P}(\text{all } N \text{ people born on different days}) =$

$$= \begin{cases} 0 & \text{if } N > 7 \\ \dfrac{\binom{7}{N}\cdot N!}{7^N} = \dfrac{(7)_N}{7^N} & \text{if } N = 2,3,4,5,6 \end{cases}$$

$\mathbb{P}(\text{at least one match}) = 1 - \frac{(7)_N}{7^N}$

Note that

$$\frac{(7)_4}{7^4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{7 \cdot 7 \cdot 7 \cdot 7} \approx 0{,}35$$

$$\frac{(7)_3}{7^3} = \frac{7 \cdot 6 \cdot 5}{7 \cdot 7 \cdot 7} \approx 0.61$$

So for $N=4 \Rightarrow \mathbb{P}(\text{at least one match}) = 1 - 0{,}35 > \frac{1}{2}$

for $N=3 \Rightarrow \mathbb{P}(\text{at least one match}) = 1 - 0{,}61 < \frac{1}{2}$

So $N=4$.

2) Define $A_{ij} = \{ i \text{ and } j \text{ are born on the same day} \}$

for $i \neq j$.

$\mathbb{P}(\text{at least one match}) = \mathbb{P}(A_{12} \cup A_{13} \cup A_{23}) =$

$= \mathbb{P}(A_{12}) + \mathbb{P}(A_{13}) + \mathbb{P}(A_{23}) - \mathbb{P}(A_{12} \cap A_{13}) - \mathbb{P}(A_{12} \cap A_{23}) +$

$- \mathbb{P}(A_{13} \cap A_{23}) + \mathbb{P}(A_{12} \cap A_{13} \cap A_{23})$

Note that for i,j,k with $i \neq j$, $i \neq k$, $j \neq k$

$$\mathbb{P}(A_{ij}) = \frac{1}{365}$$

$$\underbrace{\mathbb{P}(A_{ij} \cap A_{jk})}_{\mathbb{P}\left(\begin{array}{l}\text{3 people born on the}\\ \text{same day}\end{array}\right) = \mathbb{P}(A_{ij} \cap A_{jk} \cap A_{ik})} = \frac{1}{365^2}$$

So

$$\mathbb{P}(\text{At least one match}) = \frac{1}{365} \cdot 3 - 2 \cdot \frac{1}{365^2} =$$

$$= \frac{3 \cdot 365 - 2}{365^2} = \frac{1093}{(365)^2}$$

We got in class:

$$\mathbb{P}(\text{At least one match}) = 1 - \frac{365 \cdot 364 \cdot 363}{(365)^3} =$$

$$= \frac{365^2 - 364 \cdot 363}{(365)^2} =$$

$$= \frac{133225 - 132132}{(365)^2} =$$

$$= \frac{1093}{(365)^2}$$