EXERCISE SHEET 3

The exercise numeration aligns with the numbering system used in Chapter 3 in the book "Introduction to Probability" by David Anderson, Timo Seppalainen, and Benedek Valko.

Exercises with ** are harder and the solution is not provided. Discussions on these exercises is possible if there is time during the lectures or by asking an office hour.

1. Exercises

Ex. 1: Let X have possible values $\{1, 2, 3, 4, 5\}$ and probability mass function

$$p_X(1) = \frac{1}{7}, \qquad p_X(2) = \frac{1}{14}, \qquad p_X(3) = \frac{3}{14}, \qquad p_X(4) = p_X(5) = \frac{2}{7}.$$

Compute
(a) $\mathbb{P}(X \leq 3)$;
(b) $\mathbb{P}(X < 3)$;
(c) $\mathbb{P}(X < 4.12 \,|\, X > 1.638)$.

Ex. 3: Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 3e^{-3x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify that f is a density function.
(b) Compute $\mathbb{P}(-1 < X < 1)$.
(c) Compute $\mathbb{P}(X < 5)$.
(d) Compute $\mathbb{P}(2 < X < 4 \,|\, X < 5)$.

Ex. 7: Suppose that the continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0, & \text{if } x < \sqrt{2}, \\ x^2 - 2, & \text{if } \sqrt{2} \leq x < \sqrt{3}, \\ 1, & \text{if } \sqrt{3} \leq x. \end{cases}$$

(a) Find the smallest interval $[a, b]$ such that $\mathbb{P}(a \leq X \leq b) = 1$.
(b) Find $\mathbb{P}(X = 1.6)$.
(c) Find $\mathbb{P}\left(1 \leq X \leq \frac{3}{2}\right)$.
(d) Find the probability density function.

Ex. 9: Let X be the random variable in Ex. 3.
(a) Find the mean of X.
(b) Compute $\mathbb{E}[e^{2X}]$.

Ex. 15: Suppose the random variable X has expected value $\mathbb{E}[X] = 3$ and $\mathrm{Var}(X) = 4$. Compute the following quantities:
(a) $\mathbb{E}[3X + 2]$;
(b) $\mathbb{E}[X^2]$;
(c) $\mathbb{E}[(2X + 3)^2]$;
(d) $\mathrm{Var}(4X - 2)$.

Ex. 17: Let X be a normal random variable with mean $\mu = -2$ and variance $\sigma^2 = 7$. Find the following probabilities using the table of the standard normal distribution:
(a) $\mathbb{P}(X > 3.5)$;
(b) $\mathbb{P}(-2.1 < X < -1.9)$;
(c) $\mathbb{P}(X < 2)$;
(d) $\mathbb{P}(X < -10)$;
(e) $\mathbb{P}(X > 4)$.

Ex. 31: Suppose a random variable X has density function

$$f(x) = \begin{cases} cx^{-4}\,, \text{if } x \geq 1\,, \\ 0\,, \text{otherwise}\,, \end{cases}$$

where c is a constant.
(a) What must be the value of c?
(b) Find $\mathbb{P}(0.5 < X < 1)$.
(c) Find $\mathbb{P}(0.5 < X < 2)$.
(d) Find $\mathbb{P}(2 < X < 4)$.
(e) Find the cumulative distribution function $F_X(x)$.
(f) Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.
(g) Find $\mathbb{E}[5X^2 + 3X]$.
(h) Find $\mathbb{E}[X^n]$ for all integers n. This quantity is called the n-th moment of X. Your answer will be a formula that contains n.

Ex. 40: Give an example of a discrete or continuous random variable X (by giving the probability mass function or the probability distribution function) whose cumulative distribution function $F(x)$ satisfies $F(n) = 1 - \frac{1}{n}$ for each positive n.

Ex. 46: A stick of length ℓ is broken at a uniformly chosen random location. We denote the length of the smaller piece by X.
(a) Find the cumulative distribution function of X.
(b) Find the probability density function of X.

Ex. 52: Show that if the random variable X takes only nonnegative integers as its values, then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)\,.$$

This holds even when $\mathbb{E}[X] = \infty$, in which case the sum on the right-hand side is infinite.

Hint: Write $\mathbb{P}(X \geq k) = \sum_{i=k}^{\infty} \mathbb{P}(X = i)$ *in the sum and then switch the order of the two summations.*

Ex. 56: Let $X \sim \text{Geom}(p)$. Show that

$$\mathbb{E}[X^2] = \frac{2-p}{p^2}.$$

Hint: To compute this conveniently, start with $\mathbb{E}[X^2] = \mathbb{E}[X] + \mathbb{E}[X(X-1)]$*. Then note that in the geometric case* $\mathbb{E}[X(X-1)]$ *can be computed with the derivative technique using the second derivative* $\frac{d^2}{dx^2}x^k = k(k-1)x^{k-2}$ *for* $k \geq 2$.

Ex. 57: Let $X \sim \text{Geom}(p)$. Find the expected value of $\frac{1}{X}$.

Ex. 58: Let $X \sim \text{Binom}(n,p)$. Find the expected value of $\frac{1}{1+X}$.

Ex. 63: A random variable X is symmetric if X has the same probability distribution of $-X$. In the discrete case symmetry means $\mathbb{P}(X = k) = \mathbb{P}(X = -k)$ for all possible values k. In the continuous case it means that the density function satisfies $f(x) = f(-x)$ for all x (that is, f is an even function).
Assume that X is symmetric and $\mathbb{E}[X]$ is finite. Show that $\mathbb{E}[X] = 0$ in the
(a) discrete case;
(b) continuous case.

Ex. 69: Find a general formula for all the moments of $\mathbb{E}[Z^n]$, $n \geq 1$, for a standard normal random variable Z.

Ex. 74: Let k be a positive integer. Give an example of a nonnegative random variable X for which $\mathbb{E}[X^k] < \infty$ but $\mathbb{E}[X^{k+1}] = \infty$.

Ex. 77:** Suppose that $s_1 < s_2 < s_3 < \dots$ is an increasing sequence with $\lim_{n\to\infty} s_n = a$. Consider the events $A_n = \{X \leq s_n\}$.
(a) Show that $\cup_{n=1}^{\infty} A_n = \{X < a\}$.
(b) Prove that $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\cup_{n=1}^{\infty} A_n)$.
Hint: Recall the continuity of probability with respect to continuous monotonic sequences.

Additional Ex.: Urn U contains 2280 balls, 1000 green and 1280 red. Out of your sight, from urn U two are selected at random without replacement and put in urn A. In urn B there is one green and one red ball.
You are invited to play the following two stage game. In the first stage, you choose either urn A or urn B and withdraw a ball. If the ball is green, you are given \$10,000 and if it is red you win nothing. The ball is returned to the urn from which it is picked. In the second stage you again choose either urn A or urn B from which to randomly draw a single ball. The payoff is the same as for the first draw. How do you choose urns from which to draw in order to maximize your expected payoff?

2. Solutions

Ex. 1: Recall that the probability mass function for a discrete random variable X is defined as

$$p_X(k) = \mathbb{P}(X = k)\,.$$

(a) $\mathbb{P}(X \leq 3) = p_X(1) + p_X(2) + p_X(3) = \frac{3}{7}$.

(b) $\mathbb{P}(X < 3) = p_X(1) + p_X(2) = \frac{3}{14}$.

(c) Note that, by definition of conditional probability

$$\mathbb{P}(X < 4.12 \,|\, X > 1.638) = \frac{\mathbb{P}(X < 4.12,\, X > 1.638)}{\mathbb{P}(X > 1.638)} =$$

$$= \frac{p_X(2) + p_X(3) + p_X(4)}{p_X(2) + p_X(3) + p_X(4) + p_X(5)} = \frac{\frac{4}{7}}{\frac{6}{7}} = \frac{2}{3}\,.$$

Ex. 3: Before the computation, note that, since $f(x) = 3e^{-3x}$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$, we can rewrite $f(x)$ as

$$f(x) = 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,,$$

where

$$\mathbb{1}_{(0,+\infty)}(x) = \begin{cases} 1\,, & \text{if } x \in (0, +\infty)\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

(a) We have to check that $f(x) \geq 0$ for all $x \in \mathbb{R}$, that is obviously true, and that $\int_{-\infty}^{+\infty} f(x)\,dx = 1$. We have

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,dx =$$

$$= \int_{0}^{+\infty} 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,dx + \int_{-\infty}^{0} 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,dx =$$

$$= \int_{0}^{+\infty} 3e^{-3x} \cdot 1\,dx + \int_{-\infty}^{0} 3e^{-3x} \cdot 0\,dx =$$

$$= \int_{0}^{+\infty} 3e^{-3x}\,dx\,.$$

Recall that

$$\int e^{\alpha x}\,dx = \frac{e^{\alpha x}}{\alpha^2}\,,$$

and hence

$$\int_{0}^{+\infty} 3e^{-3x}\,dx = \left[3 \cdot \frac{e^{-3x}}{-3}\right]_0^{+\infty} = 0 - \left(e^{-3\cdot 0}\right) = 1\,.$$

So we have checked that $\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

(b) Note that

$$\mathbb{P}(-1 < X < 1) = \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-1}^{1} f(x)\,dx\,.$$

We have

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,dx\,.$$

Note that the integral is on the interval $(-1,1)$ and the function $f(x)$ is nonzero in $(0,+\infty)$. So we can reduce this integral to the interval in $(-1,1)\cap(0,+\infty)=(0,+\infty)$. Hence we have

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,dx = \int_{0}^{1} 3e^{-3x}\,dx =$$
$$= \left[3 \cdot \frac{e^{-3x}}{-3}\right]_0^1 = -e^{-3} - (-1) = 1 - e^{-3}\,.$$

(c) Note that

$$\mathbb{P}(X < 5) = \int_{-\infty}^{5} f(x)\,dx\,.$$

Since $f(x)$ is nonzero in $(0,+\infty)$ and the integral is on $(-\infty,5)$, we can reduce the above integral to the interval $(0,+\infty)\cap(-\infty,5)=(0,5)$. So

$$\int_{-\infty}^{5} f(x)\,dx = \int_{-\infty}^{5} 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,dx = \int_{0}^{5} 3e^{-3x}\,dx =$$
$$= \left[3 \cdot \frac{e^{-3x}}{-3}\right]_0^5 = -e^{-15} - (-1) = 1 - e^{-15}\,.$$

(d) By definition of conditional probability we have

$$\mathbb{P}(2 < X < 4 \,|\, X < 5) = \frac{\mathbb{P}(2 < X < 4,\, X < 5)}{\mathbb{P}(X < 5)} = \frac{\mathbb{P}(2 < X < 4)}{\mathbb{P}(X < 5)}\,.$$

We have already computed $\mathbb{P}(X < 5) = 1 - e^{-15}$. Now

$$\mathbb{P}(2 < X < 4) = \int_{2}^{4} f(x)\,dx\,.$$

Since $f(x)$ is nonzero in $(0,+\infty)$ and the integral is on $(2,4)$, we can reduce the above integral to the interval $(0,+\infty)\cap(2,4)=(2,4)$. So

$$\int_{2}^{4} f(x)\,dx = \int_{2}^{4} 3e^{-3x} \cdot \mathbb{1}_{(0,+\infty)}(x)\,dx = \int_{2}^{4} 3e^{-3x}\,dx =$$
$$= \left[3 \cdot \frac{e^{-3x}}{-3}\right]_2^4 = -e^{-12} - (-e^{-6}) = e^{-6} - e^{-12}\,.$$

So

$$\mathbb{P}(2 < X < 4 \,|\, X < 5) = \frac{\mathbb{P}(2 < X < 4,\, X < 5)}{\mathbb{P}(X < 5)} = \frac{\mathbb{P}(2 < X < 4)}{\mathbb{P}(X < 5)} = \frac{e^{-6} - e^{-12}}{1 - e^{-15}}\,.$$

Ex. 7: Recall that for any random variable X and for any $a < b$ we have

$$\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)\,.$$

Moreover, if X is continuous, then $\mathbb{P}(X = a) = 0$ for any $a \in \mathbb{R}$ and hence

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(X = a) + \mathbb{P}(a < X \leq b) = \mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)\,.$$

(a) We look for a, b such that $F_X(b) - F_X(a) = 1$. Let us assume $a < \sqrt{2}$ and $b \in [\sqrt{2}, \sqrt{3})$. Then

$$F_X(b) - F_X(a) = b^2 - 2 - 0 = b^2 - 2\,.$$

This equals 1 if

$$b^2 - 2 = 1 \Rightarrow b^2 = 3 \Rightarrow b = \pm\sqrt{3}\,.$$

Since $b \in [\sqrt{2}, \sqrt{3})$ the values we have found are not valid. So let us assume now $a < \sqrt{2}$ and $b \geq \sqrt{3}$. Then

$$F_X(b) - F_X(a) = 1 - 0 = 1\,.$$

So any interval $[a, b]$ with $a < \sqrt{2}$ and $b \geq \sqrt{3}$ works. Can we take a shorter interval? We have to check if we can think $a \in [\sqrt{2}, \sqrt{3})$ and $b \geq \sqrt{3}$. In such a case we have

$$F_X(b) - F_X(a) = 1 - (a^2 - 2) = 3 - a^2\,.$$

So $F_X(b) - F_X(a) = 1$ if $3 - a^2 = 1 \Rightarrow a^2 = 2 \Rightarrow a = \pm\sqrt{2}$. Since $a \in [\sqrt{2}, \sqrt{3})$ we can consider only $a = \sqrt{2}$. So if $a = \sqrt{2}$ and $b \geq \sqrt{3}$ we have $F_X(b) - F_X(a) = 1$. Hence the smallest interval is $[\sqrt{2}, \sqrt{3}]$.

(b) Since X is continuous we have $\mathbb{P}(X = 1.6) = 0$. We can see it also from F_X thinking that

$$\mathbb{P}(X = 1.6) = \mathbb{P}(X \in [1.6, 1.6]) = F_X(1.6) - F_X(1.6) = 0\,.$$

(c)

$$\mathbb{P}(1 \leq X \leq 3/2) = F_X(3/2) - F_X(1) = \frac{9}{4} - 2 - 0 = \frac{1}{4}\,.$$

(d) Recall that the probability density function of a continuous random variable X is the first derivative of the cumulative distribution function F_X. So

$$f(x) = \begin{cases} 2x\,, & \text{if } x \in (\sqrt{2}, \sqrt{3})\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Note that actually F_X cannot be differentiated at $x = \sqrt{2}$ and $x = \sqrt{3}$. By the way, to compute the probability we use the integral of f and it does not take care of the values of f on single points. Hence we can easily fix $f(x) = 0$ for $x = \sqrt{2}$ and $x = \sqrt{3}$.

Ex. 9: (a) Recall that if X is a continuous random variable, then

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx\,.$$

So

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx = \int_{-\infty}^{+\infty} x \cdot 3e^{-3x}\mathbb{1}_{(0,+\infty)}(x)\, dx = \int_{0}^{+\infty} x \cdot 3e^{-3x}\, dx\,.$$

Using integration by parts (in which we integrate $3e^{-3x}$ and we differentiate x), we get

$$\int_0^{+\infty} \underbrace{x}_{g(x)} \cdot \underbrace{3e^{-3x}}_{h'(x)}\, dx = \left[\underbrace{x}_{g(x)} \cdot \underbrace{(-e^{-3x})}_{h(x)}\right]_0^{+\infty} - \int_0^{+\infty} \underbrace{1}_{g'(x)} \cdot \underbrace{(-e^{-3x})}_{h(x)}\, dx =$$

$$= (0-0) - \left[\frac{-e^{-3x}}{-3}\right]_0^{+\infty} = \frac{1}{3}\,,$$

where we have used the fact that $\lim_{x\to+\infty} xe^{-3x} = 0$. So

$$\mathbb{E}[X] = \frac{1}{3}\,.$$

(b) Recall that if X is a continuous random variable, then for any function $\ell(x)$ we have

$$\mathbb{E}[\ell(X)] = \int_{-\infty}^{+\infty} \ell(x) \cdot f(x)\, dx\,.$$

So

$$\mathbb{E}[e^{2X}] = \int_{-\infty}^{+\infty} e^{2x} \cdot f(x)\, dx\,.$$

We have

$$\mathbb{E}[e^{2X}] = \int_{-\infty}^{+\infty} e^{2x} \cdot 3e^{-3x}\mathbb{1}_{(0,+\infty)}(x)\, dx = \int_0^{+\infty} 3e^{-x}\, dx = \left[3 \cdot \frac{e^{-x}}{-1}\right]_0^{+\infty} = 3(0-(-1)) = 3\,.$$

Ex. 15: (a) Recall that for any $a, b \in \mathbb{R}$ and for any two random variables X, Y we have

$$\mathbb{E}[aX+b] = a\mathbb{E}[X] + b\,, \qquad \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]\,.$$

So

$$\mathbb{E}[3X+2] = 3\mathbb{E}[X] + 2 = 3\cdot 2 + 2 = 8\,.$$

(b) Recall that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. So

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + \mathbb{E}[X]^2 = 4 + 3^2 = 13\,.$$

(c) Note that

$$(2X+3)^2 = 4X^2 + 9 + 12X\,.$$

So

$$\mathbb{E}[(2X+3)^2] = \mathbb{E}[4X^2+9+12X] = 4\mathbb{E}[X^2]+9+12\mathbb{E}[X] = 4\cdot 13+9+12\cdot 3 = 52+9+36 = 97\,.$$

(d) Recall that for any $a, b \in \mathbb{R}$ and for any random variable X we have $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$. So

$$\mathrm{Var}(4X - 2) = 16\,\mathrm{Var}(X) = 16 \cdot 4 = 64 .$$

Ex. 17: Recall that if $X \sim \mathcal{N}(\mu, \sigma^2)$ then

$$X = \sigma Z + \mu ,$$

where $Z \sim \mathcal{N}(0,1)$. Recall also that $\Phi(x) = \mathbb{P}(Z \leq x)$ is the cumulative distribution function of Z and its values are known from the table of the standard normal distribution. Hence

$$F_X(t) = \mathbb{P}(X \leq t) = \mathbb{P}(\sigma Z + \mu \leq t) = \mathbb{P}\left(Z \leq \frac{t-\mu}{\sigma}\right) = \Phi\left(\frac{t-\mu}{\sigma}\right) = \Phi\left(\frac{t+2}{\sqrt{7}}\right) .$$

Finally recall that, since the table of the standard normal distribution have the values of Φ only for $x \geq 0$, to get the values of Φ on negative numbers we can use the relation $\Phi(-x) = 1 - \Phi(x)$.

(a)

$$\mathbb{P}(X > 3.5) = 1 - \Phi\left(\frac{3.5+2}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{5.5}{\sqrt{7}}\right) \approx 1 - \Phi(2.08) \approx 1 - 0.98 = 0.02 .$$

(b)

$$\begin{aligned}\mathbb{P}(-2.1 < X < -1.9) &= \Phi\left(\frac{-1.9+2}{\sqrt{7}}\right) - \Phi\left(\frac{-2.1+2}{\sqrt{7}}\right) = \Phi\left(\frac{0.1}{\sqrt{7}}\right) - \Phi\left(-\frac{0.1}{\sqrt{7}}\right) = \\ &= \Phi\left(\frac{0.1}{\sqrt{7}}\right) - \left(1 - \Phi\left(\frac{0.1}{\sqrt{7}}\right)\right) = \\ &= 2 \cdot \Phi\left(\frac{0.1}{\sqrt{7}}\right) - 1 \approx 2\Phi(0,04) - 1 \approx 2 \cdot 0.52 - 1 = 0.04 .\end{aligned}$$

(c)

$$\mathbb{P}(X < 2) = \Phi\left(\frac{2+2}{\sqrt{7}}\right) \approx \Phi(1.51) \approx 0.93 .$$

(d)

$$\mathbb{P}(X < -10) = \Phi\left(\frac{-10+2}{\sqrt{7}}\right) \approx \Phi(-3.02) = 1 - \Phi(3.02) \approx 0 .$$

(e)

$$\mathbb{P}(X > 4) = 1 - \Phi\left(\frac{4+2}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{6}{\sqrt{7}}\right) \approx 1 - \Phi(2.27) \approx 1 - 0.99 = 0.01 .$$

Ex. 31: (a) We have to impose $\int_{-\infty}^{+\infty} f(x)\,dx = 1$. Note that

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{1}^{+\infty} cx^{-4}\,dx = c\int_{1}^{+\infty} x^{-4}\,dx = c\left[\frac{x^{-3}}{-3}\right]_1^{+\infty} = \frac{c}{3} .$$

So $\frac{c}{3} = 1 \Rightarrow c = 3$.

(b) We have

$$\mathbb{P}(0.5 < X < 1) = \int_{0.5}^{1} 3x^{-4}\mathbb{1}_{(1,+\infty)}(x)\,dx = 0\,,$$

since $(0.5, 1) \cap (1, +\infty) = \emptyset$.

(c)

$$\mathbb{P}(0.5 < X < 2) = \int_{0.5}^{2} 3x^{-4}\mathbb{1}_{(1,+\infty)}(x)\,dx = \int_{1}^{2} 3x^{-4}\,dx = [-x^{-3}]_1^2 = 1 - \frac{1}{8} = \frac{7}{8}\,.$$

(d)

$$\mathbb{P}(2 < X < 4) = \int_{2}^{4} 3x^{-4}\mathbb{1}_{(1,+\infty)}(x)\,dx = \int_{2}^{4} 3x^{-4}\,dx = [-x^{-3}]_2^4 = 1 - \frac{1}{8} = \frac{7}{8}\,.$$

(e) If $t < 1$, then

$$F_X(t) = \mathbb{P}(X \le t) = \int_{-\infty}^{t} f(x)\,dx = 0$$

since $(-\infty, t) \cap [1, +\infty) = \emptyset$ if $t < 1$. If $t \ge 1$, then

$$F_X(t) = \mathbb{P}(X \le t) = \int_{-\infty}^{t} f(x)\,dx = \int_{1}^{t} 3x^{-4}\,dx = [-x^{-3}]_1^t = 1 - \frac{1}{t^3}\,.$$

So

$$F_x(t) = \begin{cases} 1 - \frac{1}{t^3}\,, & \text{if } t \ge 1\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

(f) Note that

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx = \int_{1}^{\infty} x \cdot 3x^{-4}\,dx = \left[-\frac{3}{2}x^{-2}\right]_1^{\infty} = \frac{3}{2}\,,$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f(x)\,dx = \int_{1}^{\infty} x^2 \cdot 3x^{-4}\,dx = \left[-3x^{-1}\right]_1^{\infty} = 3\,.$$

So $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{1}{4}\,.$

(g)

$$\mathbb{E}[5X^2 + 3X] = 5\mathbb{E}[X^2] + 3\mathbb{E}[X] = 5 \cdot 3 + 3 \cdot \frac{3}{2} = \frac{39}{2}\,.$$

(h) For $n > 3$ we have

$$\mathbb{E}[X^n] = \int_{-\infty}^{+\infty} x^n \cdot f(x)\,dx = \int_{1}^{\infty} x^n \cdot 3x^{-4}\,dx = \int_{1}^{\infty} 3x^{n-4}\,dx = \left[-\frac{3}{n-3}x^{n-3}\right]_1^{\infty} = \infty\,.$$

For $n = 3$ we have

$$\mathbb{E}[X^n] = \int_{-\infty}^{+\infty} x^3 \cdot f(x)\,dx = \int_{1}^{\infty} x^n \cdot 3x^{-4}\,dx = \int_{1}^{\infty} 3x^{-1}\,dx = [3\ln(x)]_1^{\infty} = \infty\,.$$

For $n = 1, 2$ we have the results in (f).

Ex. 40: Suppose that X is a discrete random variable and assumes positive integer values. Recall that the cumulative distribution function is defined as $F(n) = \mathbb{P}(X \le n) = 1 - \frac{1}{n}$. In particular

$$\mathbb{P}(X = n) = \mathbb{P}(X \le n) - \mathbb{P}(X \le n-1) = F(n) - F(n-1) = \\ = 1 - \frac{1}{n} - \left(1 - \frac{1}{n-1}\right) = \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}\,.$$

So we can define X as a discrete random variable such that for $n \in \mathbb{N}\,, n \ge 2$ $\mathbb{P}(X = n) = \frac{1}{n(n-1)}$, otherwise $\mathbb{P}(X = n) = 0$.
Note that it is possible to show that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1\,.$

Ex. 46: Denote by Y the point at which we break the stick. Note that Y is a continuous random variable with uniform distribution on the interval $[0, \ell]$.

(a) Note that X is at most $\ell/2$ being the length of the smaller piece. In particular also X is continuous and for $t \in (0, \ell/2]$

$$F_X(t) = \mathbb{P}(X \le t) = \mathbb{P}(Y \le t) + \mathbb{P}(Y \ge \ell - t) = \\ = \mathbb{P}(Y \le t) + 1 - \mathbb{P}(Y < \ell - t) = \frac{t}{\ell} + 1 - \frac{\ell - t}{\ell} = \frac{2t}{\ell}\,.$$

So

$$F_X(t) = \begin{cases} 0\,, & \text{if } t \le 0\,, \\ \frac{2t}{\ell}\,, & \text{if } t \in (0, \ell/2]\,, \\ 1\,, & \text{if } t > \ell/2\,. \end{cases}$$

(b) Note that

$$f_X(t) = \frac{d}{dt} F_X(t) = \begin{cases} \frac{2}{\ell}\,, & \text{if } t \in (0, \ell/2)\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Ex. 52: Note that $\mathbb{P}(X \ge k) = \sum_{i=k}^{\infty} \mathbb{P}(X = i)$. Hence

$$\sum_{k=1}^{\infty} \mathbb{P}(X \ge k) = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(X = i)\,.$$

Note that we are summing over the pairs (k, i) with $k = 1, 2, \dots$ and $i \ge k$. We can rewrite such a sum considering pairs (k, i) with $i = 1, 2, \dots$ and $k \le i$. So the above sum can be rewritten as

$$\sum_{i=1}^{\infty} \sum_{k=1}^{i} \mathbb{P}(X = i) = \sum_{i=1}^{\infty} \mathbb{P}(X = i) \sum_{k=1}^{i} 1 = \sum_{i=1}^{\infty} \mathbb{P}(X = i) \cdot i = \mathbb{E}[X]\,.$$

Ex. 56: Recall that $\mathbb{P}(X = k) = p(1-p)^{k-1}$. Since

$$\mathbb{E}[X^2] = \mathbb{E}[X] + \mathbb{E}[X(X-1)] = \frac{1}{p} + \mathbb{E}[X(X-1)]$$

it is enough to compute $\mathbb{E}[X(X-1)]$. Note that

$$\mathbb{E}[X(X-1)] = \sum_{k=1}^{\infty} k(k-1)p(1-p)^{k-1} = p(1-p)\cdot \sum_{k=1}^{\infty} \frac{d^2}{dp^2}(1-p)^k =$$

$$= p(1-p)\frac{d^2}{dp^2}\sum_{k=1}^{\infty}(1-p)^k = p(1-p)\cdot \frac{d^2}{dp^2}\frac{1-p}{p} =$$

$$= p(1-p)\cdot \frac{2}{p^3} = \frac{2-2p}{p^2}\,.$$

So

$$\mathbb{E}[X^2] = \frac{1}{p} + \frac{2-2p}{p^2} = \frac{2-p}{p^2}\,.$$

Ex. 57: We have

$$\mathbb{E}\left[\frac{1}{X}\right] = \sum_{k=1}^{\infty}\frac{1}{k}\cdot p(1-p)^{k-1} = \frac{1}{1-p}\sum_{k=1}^{\infty}\frac{1}{k}\cdot p(1-p)^k\,.$$

Recall that the Taylor series of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{n+1}\frac{x^n}{n!}\,,$$

and hence

$$-\ln(1-x) = \sum_{k=1}^{\infty}\frac{x^n}{n!}\,.$$

So

$$\mathbb{E}\left[\frac{1}{X}\right] = \frac{p}{1-p}\sum_{k=1}^{\infty}\frac{(1-p)^k}{k} = \frac{p}{1-p}\cdot(-\ln(1-(1-p))) = \frac{-p\ln(p)}{1-p}\,.$$

Ex. 58: We have

$$\mathbb{E}\left[\frac{1}{X+1}\right] = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=0}^{n}\frac{n!}{(k+1)!\cdot(n-k)!}p^k(1-p)^{n-k} =$$

$$= \frac{1}{n+1}\sum_{k=0}^{n}\frac{(n+1)!}{(k+1)!\cdot(n-k)!}p^k(1-p)^{n-k} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}p^k(1-p)^{n-k} =$$

$$= \frac{1}{p(n+1)}\sum_{k=0}^{n}\binom{n+1}{k+1}p^{k+1}(1-p)^{n-k} \underset{j=k+1}{=} \frac{1}{p(n+1)}\sum_{j=1}^{n+1}\binom{n+1}{j}p^j(1-p)^{n+1-j} =$$

$$= \frac{1}{p(n+1)}\left[\sum_{j=0}^{n+1}\binom{n+1}{j}p^j(1-p)^{n+1-j} - (1-p)^{n+1}\right] = \frac{1}{p(n+1)}\left[1-(1-p)^{n+1}\right]\,,$$

where we have used the fact that

$$\sum_{j=0}^{n+1}\binom{n+1}{j}p^j(1-p)^{n+1-j} = \sum_{j=0}^{n+1}\mathbb{P}(\mathrm{Bin}(n+1,p)=j) = 1\,.$$

Ex. 63: (a) Denote by $\mathrm{Im}(X) = \{k \in \mathbb{R} \,|\, \mathbb{P}(X = k) > 0\}$. We can write $\mathrm{Im}(X) = \mathrm{Im}^+(X) \cup \mathrm{Im}^-(X)$, where

$$\mathrm{Im}^+(X) = \{k > 0 \,|\, \mathbb{P}(X = k) > 0\}\,, \qquad \mathrm{Im}^-(X) = \{k < 0 \,|\, \mathbb{P}(X = k) > 0\}\,.$$

Then

$$\mathbb{E}[X] = \sum_{k\in \mathrm{Im}(X)} k\mathbb{P}(X = k) = \sum_{k\in \mathrm{Im}^+(X)} k\mathbb{P}(X = k) + \sum_{k\in \mathrm{Im}^-(X)} k\mathbb{P}(X = k)\,.$$

Since X is symmetric we can write

$$\sum_{k\in \mathrm{Im}^-(X)} k\mathbb{P}(X = k) = \sum_{k\in \mathrm{Im}^+(X)} -k\mathbb{P}(X = -k) = -\sum_{k\in \mathrm{Im}^+(X)} k\mathbb{P}(X = k)\,.$$

Hence

$$\mathbb{E}[X] = \sum_{k\in \mathrm{Im}^+(X)} k\mathbb{P}(X = k) - \sum_{k\in \mathrm{Im}^+(X)} k\mathbb{P}(X = k) = 0\,.$$

(b) Since $f(-x) = f(x)$, we have that $f(x)$ is even. Hence the function $h(x) = xf(x)$ is odd, that is $h(x) = -h(-x)$. So

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx = \int_{-\infty}^{+\infty} h(x)\,dx = 0\,,$$

since the integral of an odd function over $\mathbb{R}$ is zero.[1].

Ex. 69: Note that

$$\mathbb{E}[Z^n] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n \cdot e^{-\frac{x^2}{2}}\,dx\,.$$

Note that

$$x^n \cdot e^{-\frac{x^2}{2}} = x^{n-1} \cdot xe^{-\frac{x^2}{2}} = -x^{n-1} \cdot \frac{d}{dx} e^{-\frac{x^2}{2}}\,.$$

So using integration by parts we have

$$\begin{aligned}\mathbb{E}[Z^n] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n \cdot e^{-\frac{x^2}{2}}\,dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^{n-1} \cdot \frac{d}{dx} e^{-\frac{x^2}{2}}\,dx = \\ &= -\frac{1}{\sqrt{2\pi}} \left(\left[x^{n-1} e^{-\frac{x^2}{2}} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (n-1)x^{n-2} e^{-\frac{x^2}{2}}\,dx \right) = \\ &= 0 + (n-1) \int_{-\infty}^{+\infty} x^{n-2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\,dx = (n-1)\mathbb{E}[Z^{n-2}]\,.\end{aligned}$$

Hence, if n is even, by iteration we get

$$\mathbb{E}[Z^n] = (n-1)I_{n-2} = (n-1)(n-3)I_{n-4} = \ldots = (n-1)(n-3)(n-5)\ldots 3\cdot 1\cdot \mathbb{E}[Z^0] = (n-1)!!\,,$$

[1] Actually this integral has meaning only in the sense of Cauchy Principal Value

where $\mathbb{E}[Z^0] = \mathbb{E}[1] = 1$ and

$$(n-1)!! = (n-1)(n-3)(n-5)\dots 3\cdot 1 = \prod_{\substack{i=1,\\ i \text{ odd}}}^{n-1} (n-i)\,.$$

If n is odd

$$\mathbb{E}[Z^n] = (n-1)I_{n-2} = (n-1)(n-3)I_{n-4} = \dots = (n-1)(n-3)(n-5)\dots 3\cdot 1\cdot \mathbb{E}[Z^1] = 0\,,$$

since $\mathbb{E}[Z] = 0$. So

$$\mathbb{E}[Z^n] = \begin{cases} (n-1)!!\,, & \text{if } n \text{ is even}\,, \\ 0\,, & \text{if } n \text{ is odd}\,, \end{cases}$$

Ex. 74: Fix a positiver integer k. Let us define a continuous random variable X with probability density function

$$f(x) = \begin{cases} \frac{c_k}{x^{k+2}}\,, & \text{if } x > 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where $c_k > 0$ is a constant dependent of k. Note that

$$\mathbb{E}[X^k] = \int_{-\infty}^{+\infty} x^k \cdot f(x)\,dx = \int_1^{+\infty} \frac{c_k}{x^2}\,dx = c_k\left[-\frac{1}{x}\right]_1^{\infty} = c_k < \infty\,.$$

Moreover

$$\mathbb{E}[X^{k+1}] = \int_{-\infty}^{+\infty} x^{k+1} \cdot f(x)\,dx = \int_1^{+\infty} \frac{c_k}{x}\,dx = c_k\left[\ln|x|\right]_1^{\infty} = \infty\,.$$

So we are left to compute c_k. Since $\int_{-\infty}^{+\infty} f(x)\,dx = 1$, we have

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{c_k}{x^{k+2}}\,dx = c_k\left[(-k-1)\frac{1}{x^{k+1}}\right]_1^{+\infty} = c_k\cdot(k+1)\,,$$

and hence

$$c_k = \frac{1}{k+1}\,.$$

So X has probability density function

$$f(x) = \begin{cases} \frac{1}{(k+1)x^{k+2}}\,, & \text{if } x > 1\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Additional Ex.: Lets say we have in general W green and B red balls, $N = W + B$. Consider the strategy of drawing first from Urn A. If the ball is green, we draw again from urn A, and if the ball is red, we take the second draw from Urn B. The possible outcomes and their probabilities are listed below. E.g. for the ww computation, inside the parentheses is the probability of picking green the second time, so with probability $1/2$ we select the green ball we chose on the first draw, and with probability 1/2 a different green ball from the original urn.

$$\begin{cases} ww \quad \text{with probability} \quad \frac{W}{N}\left(\frac{1}{2}+\frac{1}{2}\frac{W-1}{N-1}\right) = 0.3154 \\ wb \quad \text{with probability} \quad \frac{W}{N}\left(\frac{1}{2}\frac{B}{N-1}\right) = 0.1232 \\ bw \quad \text{with probability} \quad \frac{B}{N}\frac{1}{2} = 0.2807 \\ bb \quad \text{with probability} \quad \frac{B}{N}\frac{1}{2} = 0.2807 \end{cases}$$

Expected value applying this strategy is

$$20,000 * 0.3154 + 10,000 * (0.1232 + 0.2807) = 10,347.$$

Taking two balls out of Urn B gives an expected number of 1 green ball, for an expectation of 10,000.

The advantage is due to the fact that urn A has some significant chance of having two green balls, whereas urn B has no such chance.