## EXERCISE SHEET 2

The exercise numeration aligns with the numbering system used in Chapter 2 in the book "Introduction to Probability" by David Anderson, Timo Seppalainen, and Benedek Valko.

Exercises with ** are harder and the solution is not provided. Discussions on these exercises is possible if there is time during the lectures or by asking an office hour.

### 1. Exercises

**Ex. 2:** A fair coin is flipped three times. What is the probability that the second flip is tails, given that there is at most one tails among the three flips?

**Ex. 5:** We have two urns. The first urn contains three balls labeled 1, 2 and 3. The second urn contains four balls labeled 2, 3, 4, 5. We choose one of the urns randomly, so that the probability of choosing the first one is $\frac{1}{5}$ and the probability of choosing the second is $\frac{4}{5}$. Then we sample one ball (uniformly at random) from the chosen urn. What is the probability that we picked a ball labeled 2?

**Ex. 9:** In the setting of Exercise 5, suppose that ball 3 was chosen. What is the probability that it came from the second urn?

**Ex. 11:** The Acme Insurance company has two types of customers, careful and reckless. A careful customer has an accident during the year with probability 0.01. A reckless customer has an accident during the year with probability 0.04. 80% of the customers are careful and 20% of the customers are reckless. Suppose a randomly chosen customer has an accident this year. What is the probability that this customer is one of the careful customers?

**Ex. 14:** Let $A$ and $B$ be two disjoint events. Under what condition are they independent?

**Ex. 19:** We have an urn with balls labeled $1, \ldots, 7$. Two balls are drawn. Let $X_1$ be the number of the first ball drawn and $X_2$ the number of the second ball drawn. By counting favorable outcomes, compute the probabilities $\mathbb{P}(X_1 = 4)$, $\mathbb{P}(X_2 = 5)$ and $\mathbb{P}(X_1 = 4, X_2 = 5)$ in cases (a) and (b) below.
- (a) The balls are drawn one by one with replacement.
- (b) The balls are drawn one by one without replacement.
- (c) Does the answer to either (a) or (b) prove something about the independence of the random variables $X_1$ and $X_2$?

**Ex. 21:** Jane must get at least three of the four problems on the exam correct to get an $A$. She has been able to do 80% of the problems on old exams, so she assumes that the probability she gets any problem correct is 0.8. She also assumes that the results on different problems are independent.

(a) What is the probability she gets an A?
(b) If she gets the first problem correct, what is the probability she gets an A?

**Ex. 22:** Ann and Bill play rock-scissor-paper. Each one has a strategy of choosing uniformly at random out of the three possibilities every round (independently of the other player and the previous choices).

(a) What is the probability that Ann wins the first round? (Remember that the round could end in a tie).
(b) What is the probability that Ann's first win happens in the fourth round?
(c) What is the probability that Ann's first win comes after the fourth round?

**Ex. 31:** Suppose a family has two children of different ages. We assume that all combinations of boys and girls are equally likely.

(a) Formulate precisely the sample space and probability measure that describes the genders of the two children in the order in which they are born.
(b) Suppose we learn that there is a girl in the family. (Precisely: we learn that there is at least one girl.) What is the probability that the other child is a boy?
(c) Suppose we see the parents with a girl, and the parents tell us that this is their younger child. What is the probability that the older child we have not yet seen is a boy?

**Ex. 34:** You play the following game against your friend. You have two urns and three balls. One of the balls is marked. You get to place the balls in the two urns any way you please, including leaving one urn empty. Your friend will choose one urn at random and then draw a ball from that urn. (If he chooses an empty urn, there is no ball.) His goal is to draw the marked ball.

(a) How would you arrange the balls in the urns to minimize his chances of drawing the marked ball?
(b) How would you arrange the balls in the urns to maximize his chances of drawing the marked ball?
(c) Repeat (a) and (b) for the case of $n$ balls with one marked ball.

**Ex. 48:** A crime has been committed in a town of 100.000 inhabitants. The police are looking for a single perpetrator, believed to live in town. DNA evidence is found on the crime scene. Kevin's DNA matches DNA recovered from the crime scene. According to DNA experts, the probability that a random person's DNA matches the crime scene DNA is 1 in 10.000. Before the DNA evidence, Kevin was no more likely to be the guilty person than any other person in town. What is the probability that Kevin is guilty after the DNA evidence appeared?

**Ex. 67:** Show that if $X \sim \mathrm{Geom}(p)$ then

$$\mathbb{P}(X = n + k \mid X > n) = \mathbb{P}(X = k)$$

for $n, k \geq 1$. This could be called the memoryless property of the geometric distribution, because it states the following. Given that there are no successes in the first $n$ trials, the probability that the first success comes at trial $n + k$ is the same as the probability that a freshly started sequence of trials yields the first success at trial $k$. In other words, the earlier $n$ failures are forgotten. On an intuitive level this is a self-evident consequence of the independence of trials.

**Ex. 85**:** (Gambler's ruin with a fair coin) You play the following simple game of chance. A fair coin is flipped. If it comes up heads, you win a dollar. If it comes up tails, you lose a dollar. Suppose you start with $N$ dollars in your pocket. You play repeatedly until you either reach $M$ dollars or lose all your money, whichever comes first. $M$ and $N$ are fixed positive integers such that $0 < N < M$.

(a) Show that with probability one the game ends, in other words, that the amount of money in your pocket will eventually be 0 or $M$.

(b) What is the probability that the game ends with $M$ dollars in your pocket? *Hint: You can condition on the outcome of the first coin flip.*

**Ex. 88**:** We have a coin with an unknown probability $p \in (0, 1)$ for heads. Use it to produce a fair coin. You are allowed to flip the coin as many times as you wish.

## 2. Solutions

**Ex. 2:** Let us define the events

$$A = \{\text{the second flip is tails}\},$$

$$B = \{\text{there is at most one tails among the three flips}\}.$$

We have to compute $\mathbb{P}(A \mid B)$. Recall that

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Note that $\Omega = \{(x_1, x_2, x_3) \mid x_i \in \{H, T\}\}$ and hence $\#\Omega = 2 \cdot 2 \cdot 2 = 2^3 = 8$. Moreover

$$A \cap B = \{(H, T, H)\}.$$

So $\mathbb{P}(A \cap B) = \frac{1}{8}$. As far as $\mathbb{P}(B)$, note that

$$B = \{(H, H, H), (T, H, H), (H, H, T), (H, T, H)\}$$

and hence

$$\mathbb{P}(B) = \frac{4}{8} = \frac{1}{2}.$$

So

$$\mathbb{P}(A \mid B) = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}.$$

**Ex. 5:** Let us define the event $A = \{\text{the picked ball is labeled 2}\}$. To compute $\mathbb{P}(A)$ it would be useful to know from which urn we draw the ball. What we can easily say is that, defining the events

$$U_1 = \{\text{we draw from the first urn}\}, \qquad U_2 = \{\text{we draw from the second urn}\},$$

we have

$$\mathbb{P}(A \mid U_1) = \frac{1}{3}, \qquad \mathbb{P}(A \mid U_2) = \frac{1}{4}.$$

To recover $\mathbb{P}(A)$ from these two probabilities it is enough to observe that $U_1, U_2$ are a partition of the sample space. Indeed

$$U_1 \cap U_2 = \emptyset, \qquad U_1 \cup U_2 = \Omega.$$

So we can use the formula

$$\mathbb{P}(A) = \mathbb{P}(A \mid U_1)\mathbb{P}(U_1) + \mathbb{P}(A \mid U_2)\mathbb{P}(U_2) = \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{4}{5} = \frac{4}{15}.$$

**Ex. 9:** Define the event $A = \{\text{ball 3 is drawn}\}$ and

$$U_1 = \{\text{we draw from the first urn}\}, \qquad U_2 = \{\text{we draw from the second urn}\}.$$

We want to compute $\mathbb{P}(U_2 \mid A)$. We see that this conditioning is written in the opposite "direction of time" since we first choose the urn and then draw the ball. So we have to invert the conditioning with the Bayes' formula.

$$\mathbb{P}(U_2 \mid A) = \frac{\mathbb{P}(A \mid U_2)\mathbb{P}(U_2)}{\mathbb{P}(A)}.$$

Note that

$$\mathbb{P}(A) = \mathbb{P}(A \mid U_1)\mathbb{P}(U_1) + \mathbb{P}(A \mid U_2)\mathbb{P}(U_2) = \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{4}{5} = \frac{4}{15}$$

and hence

$$\mathbb{P}(U_2 \mid A) = \frac{\frac{1}{4} \cdot \frac{4}{5}}{\frac{4}{15}} = \frac{3}{4}\,.$$

**Ex. 11:** Let us define the events

$$A = \{\text{the customer has an accident}\}\,,$$
$$B = \{\text{the customer is careful}\}\,.$$

We have to compute $\mathbb{P}(B \mid A)$. We know that

$$\mathbb{P}(A \mid B) = 0.01 = \frac{1}{100}\,, \qquad \mathbb{P}(A \mid B^c) = 0.01 = \frac{4}{100}\,,$$
$$\mathbb{P}(B) = \frac{80}{100} = \frac{4}{5}\,, \qquad \mathbb{P}(B^c) = 1 - \frac{4}{5} = \frac{1}{5}\,.$$

Again, looking at our data, we have information about $\mathbb{P}(A \mid B)$ and not about $\mathbb{P}(B \mid A)$. So we have to invert the conditioning using Bayes' formula

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\mathbb{P}(B)}{\mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\mathbb{P}(B^c)} = \frac{\frac{1}{100} \cdot \frac{4}{5}}{\frac{1}{100} \cdot \frac{4}{5} + \frac{4}{100} \cdot \frac{1}{5}} = \frac{\frac{1}{125}}{\frac{2}{125}} = \frac{1}{2}\,.$$

**Ex. 14:** We know that $A$ and $B$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\,.$$

Since by hypothesis $A \cap B = \emptyset$, the above formula gives

$$0 = \mathbb{P}(A)\mathbb{P}(B)\,,$$

that implies $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$. So two disjoint events are independent if and only if at least one of the two events have probability zero.

**Ex. 19:** (a) Since the balls are drawn with replacement at each round the urn as the same composition. So

$$\mathbb{P}(X_1 = 4) = \mathbb{P}(X_2 = 5) = \frac{1}{7}\,.$$

Moreover

$$\mathbb{P}(X_1 = 4,\, X_2 = 5) = \mathbb{P}(X_2 = 5 \mid X_1 = 4)\mathbb{P}(X_1 = 4)\,.$$

Since the composition of the urn is the same at each round, the outcome of the first drawing does not affect the outcome of the second drawing. Hence the events $\{X_1 = 4\}$ and $\{X_2 = 5\}$ are independent and

$$\mathbb{P}(X_2 = 5 \mid X_1 = 4) = \mathbb{P}(X_2 = 5) = \frac{1}{7}\,.$$

So

$$\mathbb{P}(X_1 = 4,\, X_2 = 5) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}\,.$$

(b) In this case, since balls are drawn without replacement, we have the configuration of the urn at the second round changes after the first drawing. So we have $\mathbb{P}(X_1 = 4) = \frac{1}{7}$ and

$$\mathbb{P}(X_2 = 5) = \mathbb{P}(X_2 = 5 \,|\, X_1 = 5)\mathbb{P}(X_1 = 5) + \mathbb{P}(X_2 = 5 \,|\, X_1 \neq 5)\mathbb{P}(X_1 \neq 5) =$$
$$= 0 \cdot \frac{1}{7} + \frac{1}{6} \cdot \frac{6}{7} = \frac{1}{7}\,.$$

Moreover

$$\mathbb{P}(X_1 = 4, X_2 = 5) = \mathbb{P}(X_2 = 5 \,|\, X_1 = 4)\mathbb{P}(X_1 = 4) = \frac{1}{6} \cdot \frac{1}{7} = \frac{1}{42}\,.$$

So the events $\{X_1 = 4\}$ and $\{X_2 = 5\}$ are not independent being

$$\mathbb{P}(X_1 = 4, X_2 = 5) = \frac{1}{42} \neq \frac{1}{7} \cdot \frac{1}{7} = \mathbb{P}(X_1 = 4)\mathbb{P}(X_2 = 5)\,.$$

(c) In case (a) we have independence, while in case (b) we do not have it.

**Ex. 21:** (a) Define the event $A = \{\text{Jane takes grade A at the exam}\}$. Let us see each exercise as an independent experiment with probability of success $\frac{8}{10}$. Let us denote by $X$ the number of right exercises and observe that $X$ can assume values 0, 1, 2, 3, 4. We see that the event $A$ coincide with the event $\{X \geq 3\}$. Note that

$$\mathbb{P}(X = 4) = \left(\frac{8}{10}\right)^4,$$

since we have 4 right exercises, and

$$\mathbb{P}(X = 3) = \binom{4}{3}\left(\frac{8}{10}\right)^3 \frac{2}{10},$$

since we have 3 right exercises and we have to decide which of the 4 exercises are these 3 right exercises (this justifies the binomial coefficient). Note that the above formulas can be justified by saying that $X \sim \text{Bin}\left(4, \frac{8}{10}\right)$.

So

$$\mathbb{P}(X \geq 3) = \mathbb{P}(X = 3) + \mathbb{P}(X = 4) = \binom{4}{3}\left(\frac{8}{10}\right)^3 \frac{2}{10} + \left(\frac{8}{10}\right)^4 =$$
$$= \left(\frac{8}{10}\right)^3 \left[4 \cdot \frac{2}{10} + \frac{8}{10}\right] = \left(\frac{8}{10}\right)^4 \cdot 2 \approx 0.82\,.$$

(b) We have to compute $\mathbb{P}(Y \geq 2)$, where $Y$ is the number of right exercises among three exercises. Since the probability that an exercise is right is $\frac{8}{10}$ and since $Y$ can assume values 0, 1, 2, 3, we have that

$$\mathbb{P}(Y \geq 2) = \mathbb{P}(Y = 2) + \mathbb{P}(Y = 3) = \binom{3}{2}\left(\frac{8}{10}\right)^2 \frac{2}{10} + \left(\frac{8}{10}\right)^3 =$$
$$= \left(\frac{8}{10}\right)^2 \left[3 \cdot \frac{2}{10} + \frac{8}{10}\right] = \left(\frac{8}{10}\right)^2 \cdot \frac{14}{10} \approx 0.9\,.$$

**Ex. 22:** Let us denote by $(x, y)$ the pair of strategies used by Ann and Bob respectively, that is Ann plays $x$ and Bob plays $y$, where $x, y \in \{R, S, P\}$ (where $R, S, P$ stands for rock, scissor, paper, respectively).
If we look at one round, the sample space of the experiment is made by all possible pair of strategies. So $\#\Omega = 3 \cdot 3 = 3^2 = 9$.

(a) Define the event

$$A = \{\text{Ann win at the first round}\} = \{(R, S), (S, P), (P, R)\}.$$

Then

$$\mathbb{P}(A) = \frac{\#A}{\#O} = \frac{3}{9} = \frac{1}{3}.$$

(b) All rounds are independent and with winning probability $\frac{1}{3}$ (by point (a)). So

$$\mathbb{P}(\text{Ann's first win happens in the fourth round}) = \left(\frac{2}{3}\right)^3 \frac{1}{3} = \frac{8}{81},$$

since Ann looses for 3 times and wins at the fourth round.

(c) Note that

$$\mathbb{P}(\text{Ann's first win happens after the fourth round}) = \left(\frac{2}{3}\right)^4 = \frac{16}{81},$$

since Ann's first win happens after the fourth round if Ann has not won the first four rounds.

**Ex. 31:** (a) The sample space $\Omega$ is made by all pairs $(x, y)$, where $x, y \in \{G, B\}$ are, respectively, the genders of the first and second child (where $G$ stands for girl and $B$ stands for boy). So

$$\Omega = \{(G, G), (G, B), (B, G), (B, B)\}$$

where all outcomes are equally likely and hence each outcome has probability to occur $\frac{1}{4}$.

(b) We are asking the probability that one child is a boy knowing that there is at least one girl. Let us define the events

$$A = \{\text{there is at least a girl}\}, \qquad B = \{\text{one child is a boy}\}.$$

We want to compute $\mathbb{P}(B \mid A)$. So

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}.$$

Note that

$$B \cap A = \{(B, G), (G, B)\},$$

and hence $\mathbb{P}(B \cap A) = \frac{2}{4} = \frac{1}{2}$. Moreover

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \mathbb{P}(\{(B, B)\}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

So

$$\mathbb{P}(B \mid A) = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}\,.$$

(c) Let us define the events

$$A = \{\text{the second child is a boy}\}\,, \qquad B = \{\text{the first child is a boy}\}\,.$$

We want to compute $\mathbb{P}(B \mid A)$. Note that $B \cap A = \{(B, B)\}$ and hence $\mathbb{P}(B \cap A) = \frac{1}{4}$. Moreover $A = \{(B, B), (G, B)\}$ and hence $\mathbb{P}(A) = \frac{1}{2}$. So

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}\,.$$

**Ex. 34:** Let us define the following events

$$U_1 = \{\text{your friend draws from urn 1}\}\,, \qquad U_2 = \{\text{your friend draws from urn 2}\}\,,$$

$$A = \{\text{you draw the marked ball}\}\,.$$

We have to compute $\mathbb{P}(A)$ in three different situations:

(i) urn 1 has 3 balls;
(ii) urn 1 has 2 balls of which 1 is marked;
(iii) urn 1 has 2 non-marked balls.

In case (i)

$$\mathbb{P}(A) = \mathbb{P}(A \mid U_1)\mathbb{P}(U_1) + \mathbb{P}(A \mid U_2)\mathbb{P}(U_2) = \frac{1}{3} \cdot \frac{1}{2} + 0 = \frac{1}{6}\,.$$

In case (ii)

$$\mathbb{P}(A) = \mathbb{P}(A \mid U_1)\mathbb{P}(U_1) + \mathbb{P}(A \mid U_2)\mathbb{P}(U_2) = \frac{1}{2} \cdot \frac{1}{2} + 0 = \frac{1}{4}\,.$$

In case (iii)

$$\mathbb{P}(A) = \mathbb{P}(A \mid U_1)\mathbb{P}(U_1) + \mathbb{P}(A \mid U_2)\mathbb{P}(U_2) = 0 + 1 \cdot \frac{1}{2} = \frac{1}{2}\,.$$

So we have the following conclusions:

(a) to minimize the probability of drawing the marked ball we have to place all balls in one urn;
(b) to maximize the probability of drawing the marked ball we have to place the marked ball alone in one urn;
(c) let us distinguish between these cases
  (i) urn 1 has n balls;
  (ii) urn 1 has x (with $1 \leq x < n$) balls of which 1 is marked.

In case (i)

$$\mathbb{P}(A) = \mathbb{P}(A \mid U_1)\mathbb{P}(U_1) + \mathbb{P}(A \mid U_2)\mathbb{P}(U_2) = \frac{1}{n} \cdot \frac{1}{2} + 0 = \frac{1}{2n}\,.$$

In case (ii)

$$\mathbb{P}(A) = \mathbb{P}(A \mid U_1)\mathbb{P}(U_1) + \mathbb{P}(A \mid U_2)\mathbb{P}(U_2) = \frac{1}{x} \cdot \frac{1}{2} + 0 = \frac{1}{2x}\,.$$

Note that the probability is minimal in case (i) and maximal in case (ii) when $x = 1$. So, as for the case $n = 3$ we have that to minimize the probability of drawing the marked ball we have to place all balls in one urn, while to maximize the probability of drawing the marked ball we have to place the marked ball alone in one urn.

**Ex. 48:** Let us define the events $A = \{\text{Kevin is guilty}\}$ and $B = \{\text{DNA match}\}$. We know that

$$\mathbb{P}(A) = \frac{1}{100.000}, \qquad \mathbb{P}(B \mid A^c) = \frac{1}{10.000}.$$

We want to compute $\mathbb{P}(A \mid B)$. Note that $\mathbb{P}(B \mid A) = 1$ and hence by Bayes formula we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c)} =$$

$$= \frac{\frac{1}{10^5}}{\frac{1}{10^5} + \frac{1}{10^4} \cdot \left(1 - \frac{1}{10^5}\right)} = \frac{1}{1 + 10 - \frac{1}{10^4}} \approx \frac{1}{11}.$$

**Ex. 67:** Recall that $X \sim \mathrm{Geom}(p)$ if $X$ denotes the first time we get a success by repeating the same experiment independently many times, where each time the probability that the experiment is a success is $p$.

Note that

$$\mathbb{P}(X = n + k \mid X > n) = \frac{\mathbb{P}(X = n + k,\, X > n)}{\mathbb{P}(X > n)} \overset{k \geq 1}{=} \frac{\mathbb{P}(X = n + k)}{\mathbb{P}(X > n)}.$$

Since $X > n$ means that we have that the first $n$ experiments are not success, then

$$\mathbb{P}(X > n) = (1 - p)^n.$$

Moreover $X = n + k$ means that we get the first success at round $n + k$. So

$$\mathbb{P}(X = n + k) = (1 - p)^{n+k-1} p.$$

Hence

$$\mathbb{P}(X = n + k \mid X > n) = \frac{(1 - p)^{n+k-1} p}{(1 - p)^n} = (1 - p)^{k-1} p.$$

Note that $X = k$ means that the first success is at round $k$. So

$$\mathbb{P}(X = k) = (1 - p)^{k-1} p.$$

Hence we conclude that

$$\mathbb{P}(X = n + k \mid X > n) = \mathbb{P}(X = k).$$