EXERCISE SHEET 1

The exercise numeration aligns with the numbering system used in Chapter 1 in the book "Introduction to Probability" by David Anderson, Timo Seppalainen, and Benedek Valko.

Exercises with ** are harder and the solution is not provided. Discussions on these exercises is possible if there is time during the lectures or by asking an office hour.

1. Exercises

Ex. 1: We roll a fair die twice. Describe a sample space Ω and a probability measure $\mathbb{P}$ to model this experiment. Let A be the event that the second roll is larger than the first. Find the probability $\mathbb{P}(A)$ that the event A occurs.

Ex. 2: For breakfast Bob has three options: cereal, eggs or fruit. He has to choose exactly two items out of three available.

(a) Describe the sample space of this experiment.
Hint: What are the different possible outcomes for Bob's breakfast?

(b) Let A be the event that Bob's breakfast includes cereal. Express A as a subset of the sample space.

Ex. 8: Suppose a bag of scrabble tiles contains 5 Es, 4 As, 3 Ns and 2 Bs. It is my turn and I draw 4 tiles from the bag without replacement. Assume that my draw is uniformly random. Let C be the event that I got two Es, one A and one N.

(a) Compute $\mathbb{P}(C)$ by imagining that the tiles are drawn one by one as an ordered sample.

(b) Compute $\mathbb{P}(C)$ by imagining that the tiles are drawn all at once as an unordered sample.

Ex. 9: We break a stick at a uniformly chosen random location. Find the probability that the shorter piece is less than $\frac{1}{5}$-th of the original.

Ex. 12: We roll a fair die repeatedly until we see the number four appear and then we stop.

(a) What is the probability that we need at most 3 rolls?

(b) What is the probability that we needed an even number of die rolls?
Hint: you have to use the geometric sum. Fix $r \in \mathbb{R}$ such that $-1 < r < 1$. Then for any $a \in \mathbb{N}$ we have

$$\sum_{k=a}^{\infty} r^k = \frac{r^a}{1-r}\,.$$

Ex. 13: At a certain school, 25% of the students wear a watch and 30% wear a bracelet. Moreover, 60% of the students wear neither a watch nor a bracelet. One of the students is chosen uniformly at random.
(a) What is the probability that the student is wearing a watch or a bracelet?
(b) What is the probability that the student is wearing both a watch and a bracelet?

Ex. 16: We flip a fair coin five times. For every heads you pay me \$1 and for every tails I pay you \$1. Let X denote my net winnings at the end of five flips. Find the possible values and the probability mass function of X.

Ex. 19: You throw a dart and it lands uniformly at random on a circular dartboard of radius 6 inches. If your dart gets to within 2 inches of the center I will reward you with 5 dollars. But if your dart lands farther than 2 inches away from the center I will give you 1 dollar. Let X denote the amount of your reward in dollars. Find the possible values and the probability mass function of X.

Ex. 21: Suppose an urn contains three black chips, two red chips and two green chips. We draw three chips without replacement. Let A be the event that all three chips are of different color.
(a) Compute $\mathbb{P}(A)$ by imagining that the chips are drawn one by one as an ordered sample.
(b) Compute $\mathbb{P}(A)$ by imagining that the chips are drawn all at once as an unordered sample.

Ex. 23: The Monty Hall problem is a famous math problem loosely based on a television quiz show hosted by Monty Hall. You (the contestant) face three closed doors. There is a big prize behind one door out of three, but you cannot yet see what is behind it. Monty opens one of the two other doors to reveal that there is nothing behind it. Then he offers you one chance to switch your choice. Should you switch?
This question has generated a great deal of confusion. An intuitively appealing quick answer goes like this. After Monty showed you a door with no prize, the prize is equally likely to be behind either one of the other two doors. Thus switching makes no difference. Is this correct?
To get clarity, let us use the simplest possible model. Let the outcome of the experiment be the door behind which the prize hides. So the sample space is $\Omega = \{1, 2, 3\}$. Assume that the prize is equally likely to be behind any one of the doors, and so we set $\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \mathbb{P}(\{3\}) = \frac{1}{3}$. Your decision to switch or not is not part of the model. Rather, use the model to compute the probability of winning separately for the two scenarios. We can fix the labeling of the doors so that your initial choice is door number 1. In both parts (a) and (b) below determine, for each door i, whether you win or not when the prize is behind door i.
(a) Suppose you will not switch. What is the probability that you win the prize?

(b) Suppose you will switch after Monty shows a door without a prize. What is the probability that you win the prize?

Ex. 30: Eight rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture any of the other rooks? Translation for those who are not familiar with chess: pick 8 unit squares at random from an 8×8 square grid. What is the probability that no two chosen squares share a row or a column?

Ex. 32: You are dealt five cards from a standard deck of 52. Find the probability of being dealt a full house. (This means that you have three cards of one face value and two cards of a different face value. An example would be three queens and two fours.)

Ex. 41: Imagine a game of three players where exactly one player wins in the end and all players have equal chances of being the winner. The game is repeated four times. Find the probability that there is at least one person who wins no game.

Ex. 43: Show that for any events $A_1, \ldots, A_n$ we have

$$\mathbb{P}(A_1 \cup \ldots \cup A_n) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)\,.$$

Ex. 51:** A monkey sits down in front of a computer and starts hitting the keys of the keyboard randomly. Somebody left open a text editor with an empty file, so everything the monkey types is preserved in the file. Assume that each keystroke is a uniform random choice among the entire keyboard and does not depend on the previous keystrokes. Assume also that the monkey never stops. Show that with probability one the file will eventually contain your favorite novel in full. (This means that the complete text of the novel appears from some keystroke onwards).
Assume for simplicity that each symbol (letter and punctuation mark) needed for writing the novel can be obtained by a single keystroke on the computer.

Ex. 58:** Suppose n people arrive for a show and leave their hats in the cloakroom. Unfortunately, the cloakroom attendant mixes up the hats completely so that each person leaves with a random hat. Let us assume that all $n!$ assignments of hats are equally likely. Find the probability that exactly ℓ people out of n receive the correct hat, and find the limit of this probability as $n \to \infty$.
Hint: the following identity holds

$$\#\text{ arrangements where no person among } n \text{ gets the correct hat} = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}\,.$$

Find the probability that individuals $i_1, i_2, \ldots, i_\ell$ and nobody else gets the correct hat. Add over all $i_1 < i_2 < \ldots < i_\ell$. For the limit you should get

$$\lim_{n\to\infty} \mathbb{P}(\text{exactly } \ell \text{ people out of } n \text{ receive the correct hat}) = \frac{e^{-1}}{\ell!}.$$

Ex. 59:** (Buffon's needle problem) Suppose that we have an infinite grid of parallel lines on the plane, spaced one unit apart. (For example, on the xy-plane take the lines $y = n$ for all integers n). We also have a needle of length ℓ, with $\ell \in (0, 1)$. We drop the needle on the grid and it lands in a random position. Show that the probability that the needle does not intersect any of the parallel lines is $\frac{2\ell}{\pi}$.

Hint: It is important to set up a probability space for the problem before attempting the solution. Suppose the grid consists of the horizontal lines $y = n$ for all integers n. The needle is a line segment of length ℓ. To check whether the needle intersects one of the parallel lines we need the following two quantities (see Figure 1):
- *the distance d from the center of the needle to the nearest parallel line below,*
- *the angle α of the needle measured from the horizontal direction.*

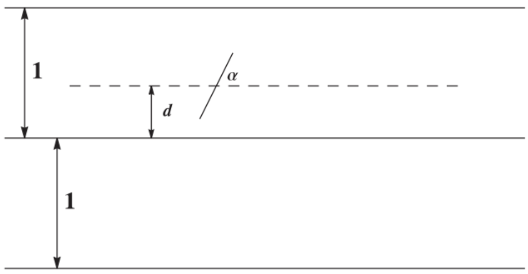

FIGURE 1

We have $d \in [0, 1)$ and $\alpha \in [0, \pi)$. We may assume (although the problem did not state it explicitly) that none of the possible pairs (d, α) is any likelier than another. This means that we can model the random position of the needle as a uniformly chosen point (d, α) from the rectangle $[0, 1) \times [0, \pi)$. To finish the proof complete the following two steps.

(a) *Describe the event $A = \{$the needle does not intersect any of the lines$\}$ in terms of d and α. You need a little bit of trigonometry.*

(b) *Compute $\mathbb{P}(A)$ as a ratio of areas.*

This problem was first introduced and solved by Georges Louis Leclerc, Comte de Buffon, in 1777.

2. Solutions

Ex. 1: We can write Ω as the set of pairs (a, b) where a and b represent the outcome of the first and second die, respectively. So

$$\Omega = \{(a, b) \mid a, b = 1, 2, 3, 4, 5, 6\}\,.$$

Alternatively we can see Ω as the set made by all entries of the following 6×6 matrix M

$$M = \begin{pmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{pmatrix}.$$

The event

$$A = \{\text{the second roll is larger than the first}\}$$

can be written as the subset of Ω made by all entries of the matrix M that are above the diagonal, that is the entries in red

$$M = \begin{pmatrix} (1,1) & \color{red}{(1,2)} & \color{red}{(1,3)} & \color{red}{(1,4)} & \color{red}{(1,5)} & \color{red}{(1,6)} \\ (2,1) & (2,2) & \color{red}{(2,3)} & \color{red}{(2,4)} & \color{red}{(2,5)} & \color{red}{(2,6)} \\ (3,1) & (3,2) & (3,3) & \color{red}{(3,4)} & \color{red}{(3,5)} & \color{red}{(3,6)} \\ (4,1) & (4,2) & (4,3) & (4,4) & \color{red}{(4,5)} & \color{red}{(4,6)} \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \color{red}{(5,6)} \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{pmatrix}.$$

Since all outcomes in Ω are equally likely, we can simply use the formula

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega}\,,$$

where $\#A$ is the number of elements of A. Since Ω is made by 36 elements (the number of entries of M) while A is made by 15 elements (the red entries in M), we have that

$$\mathbb{P}(A) = \frac{15}{36} = \frac{5}{12}\,.$$

Ex. 2: Let us denote by C, E, F, respectively, cereals, eggs and fruit.

(a) The sample space is by all non-ordered couples made by two items chosen from $\{C, E, F\}$, that is

$$\Omega = \{(C, E), (C, F), (E, F)\}\,.$$

(b) Let us consider the event $A = \{\text{Bob's breakfast includes cereals}\}$. Then

$$A = \{(C, E), (C, F)\} \subset \Omega\,.$$

Ex. 8: Define

$$C = \{\text{we have E,E,A,N}\}\,.$$

To make the computation easy let us think that all the tiles are different, that is the bag is of the following form

$$\mathcal{B} = \{E_1, \dots, E_5, A_1, \dots, A_4, N_1, \dots, N_3, B_1, B_2\}\,.$$

Note that in total we have 14 letters.

(a) Since the tiles are drawn one by one, the sample space Ω is made by all permutations of 4 symbols drawn from the bag $\mathcal{B}$. This is equal to

$$\#\Omega = (14)_4 = 14 \cdot 13 \cdot 12 \cdot 11\,.$$

The event C instead is made by all permutations of E_i, E_j, A_k, N_ℓ, where E_i, E_j (with $i \neq j$) are chosen from $E_1, \dots, E_5$, A_k is chosen from $A_1, \dots, A_4$ and N_ℓ is chosen from N_1, N_2, N_3.
Let us first count the total number of ways to choose these letters.
We have $\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$ ways to choose E_i, E_j (since we have to choose two symbols from 5 and the order of this two symbols is not relevant now). We have 4 ways to choose A_k and 3 ways for N_ℓ.
So we have $10 \cdot 4 \cdot 3$ ways in total to choose E_i, E_j, A_k, N_ℓ.
After having fixed the letters that appear, to complete the computation we have to consider all the possible orders in which we may see them.
Since we have 4 different symbols, the number of ways in which we can arrange them is 4!. Hence we conclude that

$$\#C = 4! \cdot \binom{5}{2} \cdot 4 \cdot 3\,.$$

Hence

$$\mathbb{P}(C) = \frac{\#C}{\#\Omega} = \frac{4! \cdot 10 \cdot 4 \cdot 3}{14 \cdot 13 \cdot 12 \cdot 11} = \frac{120}{1001}\,.$$

(b) Since the tiles are drawn all at once, the sample space Ω is made by all the possible ways to choose 4 symbols from the bag $\mathcal{B}$ (the order does not count). So we have

$$\#\Omega = \binom{14}{4} = \frac{14!}{4! \cdot 10!} = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4!} = 1001\,.$$

The event C instead is made by all the sequences of the form E_i, E_j, A_k, N_ℓ, where E_i, E_j (with $i \neq j$) are chosen from $E_1, \dots, E_5$, A_k is chosen from $A_1, \dots, A_4$ and N_ℓ is chosen from N_1, N_2, N_3. So we have

$$\#C = \binom{5}{2} \cdot \binom{4}{1}\binom{3}{1} = 10 \cdot 4 \cdot 3 = 120\ ,$$

since we take 2 tiles from the 5 Es, 1 tile from the 4 As, 1 tile from the 3 Ns (note that this is exactly the same computation of point (a) without the 4! since, being the four letters drawn all at once, we do not count all the permutations of the letters. Hence

$$\mathbb{P}(C) = \frac{120}{1001}\,.$$

Note that the results of (a) and (b) coincide. From this we deduce that, if the event of which we are computing the probability does not make any reference to specific order of outcomes, then we can choose whether or not order matters when we compute such a probability.

Ex. 9: Let us model the stick as the interval $I = [0, L]$ and let us denote by X the point at which we break the stick. Such a point is chosen uniformly at random. If we get a piece of stick whose length is less than $\frac{1}{5}$-th of the original (that is L), we need X to be inside the interval $\left[0, \frac{L}{5}\right]$ or $\left[\frac{4}{5}L, L\right]$. Indeed in the first case the piece $[0, X]$ will have the required length, while in the second case the piece $[X, 1]$ will solve the problem. So we are left to compute the probability that a point X taken uniformly at random from $I = [0, L]$, lies in the region $A = \left[0, \frac{L}{5}\right] \cup \left[\frac{4}{5}L, L\right]$. Since the length of I is L and the total length of A is $\frac{L}{5} + \frac{L}{5} = \frac{2L}{5}$, we have that the required probability is

$$\frac{\frac{2L}{5}}{L} = \frac{2}{5}\,.$$

Obviously such a probability does not depend on L.

Ex. 12: (a) Let us define the event $A = \{\text{we need at most 3 rolls}\}$. It is easier to analyze the complementary event

$$A^c = \{\text{we need more than 3 rolls}\} = \{\text{the first three rolls are different from 4}\}\,.$$

To analyze A^c we can reduce the analysis to the experiment with 3 rolls. In such a case Ω is made by the the vectors (x_1, x_2, x_3) where $x_i \in \{1, 2, 3, 4, 5, 6\}$ is the result of the i-th roll. Since each x_i can assume 6 values, we have that Ω is made by $6 \cdot 6 \cdot 6 = 6^3$ elements.
As far as A^c, such an event is made by vectors (x_1, x_2, x_3) where $x_i \in \{1, 2, 3, 4, 5, 6\}$. Hence A^c is made by $5 \cdot 5 \cdot 5 = 5^3$ elements. So[1]

$$\mathbb{P}(A^c) = \frac{\#A^c}{\#\Omega} = \frac{5^3}{6^3} = \left(\frac{5}{6}\right)^3\,.$$

Then we have

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}\,.$$

(b) Fix n a positive integer. Let us define the event

$$A_n = \{\text{we need } n \text{ rolls}\}\,.$$

[1] This result can be obtained by thinking that $\frac{5}{6}$ is the probability to get a number different from 4 with one die, and hence $\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{5}{6}\right)^3$ is the probability to get for three times an outcome different from 4.

Note that A is made by vectors $(x_1, x_2, \dots, x_{n-1}, 4)$, where $x_1, \dots, x_{n-1} \in \{1, 2, 3, 5, 6\}$. So

$$\#A_n = \underbrace{5 \cdot 5 \cdot \ldots \cdot 5}_{n-1 \text{ times}} = 5^{n-1}\,.$$

As far as $\#\Omega$ in this case we have that it is composed by vectors $(x_1, x_2, \dots, x_{n-1}, x_n)$, where $x_1, \dots, x_n \in \{1, 2, 3, 5, 6\}$. So

$$\#\Omega = \underbrace{6 \cdot 6 \cdot \ldots \cdot 6}_{n \text{ times}} = 6^n\,.$$

Hence[2]

$$\mathbb{P}(A_n) = \frac{5^{n-1}}{6^n} = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}\,.$$

Now let n be an even number, that is $n = 2k$ for $k \in \mathbb{N}_{>0}$. Then

$$\begin{aligned}\mathbb{P}(\{\text{we need an even number of rounds}\}) &= \sum_{k=1}^{\infty} \mathbb{P}(A_{2k}) = \\ &= \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k-1} \cdot \frac{1}{6} = \frac{1}{6} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k-1} = \\ &= \frac{1}{6} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k} \cdot \frac{6}{5} = \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k} = \\ &= \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{25}{36}\right)^{k}\,.\end{aligned}$$

Since $\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$ for any $r \in \mathbb{R}$ such that $|r| < 1$, we have

$$\frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{25}{36}\right)^{k} = \frac{1}{5} \cdot \left(\frac{\frac{25}{36}}{1 - \frac{25}{36}}\right) = \frac{1}{5} \cdot \frac{25}{36} \cdot \frac{36}{11} = \frac{5}{11}\,.$$

So

$$\mathbb{P}(\{\text{we need an even number of rounds}\}) = \frac{5}{11}\,.$$

Ex. 13: Let us define the events

$$W = \{\text{the student wears a watch}\}\,, \qquad B = \{\text{the student wears a bracelet}\}\,.$$

We know that

$$\mathbb{P}(W) = \frac{25}{100} = \frac{1}{4}\,, \qquad \mathbb{P}(B) = \frac{30}{100} = \frac{3}{10}\,,$$

$$\mathbb{P}(W^c \cap B^c) = \frac{60}{100} = \frac{3}{5}\,.$$

[2] This result can be obtained by thinking that $\frac{5}{6}$ is the probability to get a number different from 4 with one die, and hence $\left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$ is the probability to get for $n-1$ times an outcome different from 4 and to get finally a 4 at the n-th rolling.

(a) We have to compute $\mathbb{P}(W \cup B)$. By De Morgan's Law we have

$$W^c \cap B^c = (W \cup B)^c .$$

So

$$\mathbb{P}(W \cup B) = 1 - \mathbb{P}((W \cup B)^c) = 1 - \mathbb{P}(W^c \cap B^c) = 1 - \frac{3}{5} = \frac{2}{5} .$$

(b) By the inclusion/exclusion formula we have

$$\mathbb{P}(W \cap B) = \mathbb{P}(W) + \mathbb{P}(B) - \mathbb{P}(W \cup B) = \frac{1}{4} + \frac{3}{10} - \frac{2}{5} = \frac{3}{20} .$$

Ex. 16: Let us define a vector $(X_1, \dots, X_5)$ where for $i = 1, \dots, 5$

$$X_i = \begin{cases} 1 , & \text{if we get head at the } i\text{-th flip} , \\ -1 , & \text{if we get tail at the } i\text{-th flip} . \end{cases}$$

So our net gain is $X = \sum_{i=1}^{5} X_i$. Note that

$$X = \begin{cases} 5 , & \text{if we get 5 heads and 0 tails} , \\ 3 , & \text{if we get 4 heads and 1 tails} , \\ 1 , & \text{if we get 3 heads and 2 tails} , \\ -1 , & \text{if we get 2 heads and 3 tails} , \\ -3 , & \text{if we get 1 heads and 4 tails} , \\ -5 , & \text{if we get 0 heads and 5 tails} . \end{cases}$$

To compute the probabilities of all the events listed above, note that the sample space Ω is made by all vectors $(X_1, \dots, X_5)$ where $X_i \in \{-1, 1\}$. So

$$\#\Omega = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 .$$

As a consequence

$$\mathbb{P}(\text{we get 5 heads}) = \mathbb{P}((1,1,1,1,1)) = \frac{1}{2^5} = \frac{1}{32} .$$

Similarly

$$\mathbb{P}(\text{we get 5 tails}) = \mathbb{P}((-1,-1,-1,-1,-1)) = \frac{1}{2^5} = \frac{1}{32} .$$

Note that we get 4 heads and 1 tails if the outcome is a permutation of $(1, 1, 1, 1, -1)$. To count all these permutations we have to count all the ways to choose 4 entries among 5 that we will assign as 1. Such a number is given by $\binom{5}{4} = 5$ (note we are not interested in the order of the entries we have chosen). So

$$\mathbb{P}(\text{we get 4 heads and 1 tails}) = \frac{\binom{5}{4}}{2^5} = \frac{5}{32} .$$

Similarly

$$\mathbb{P}(\text{we get 1 heads and 4 tails}) = \frac{\binom{5}{4}}{2^5} = \frac{5}{32} .$$

Note that we get 3 heads and 2 tails if the outcome is a permutation of $(1,1,1,-1,-1)$. To count all these permutations we have to count all the ways to choose 3 entries among 5 that we will assign as 1. Such a number is given by $\binom{5}{3} = 10$ (note we are not interested in the order of the entries we have chosen). So

$$\mathbb{P}(\text{we get 3 heads and 2 tails}) = \frac{\binom{5}{3}}{2^5} = \frac{10}{32} = \frac{5}{16}\,.$$

Similarly

$$\mathbb{P}(\text{we get 2 heads and 3 tails}) = \frac{\binom{5}{3}}{2^5} = \frac{5}{16}\,.$$

So the probability mass of X is

$$\mathbb{P}(X=5) = \mathbb{P}(X=-5) = \frac{1}{32}\,, \qquad \mathbb{P}(X=3) = \mathbb{P}(X=-3) = \frac{5}{32}\,,$$

$$\mathbb{P}(X=1) = \mathbb{P}(X=-1) = \frac{5}{16}\,.$$

Ex. 19: The area of a circle of radius r is πr^2. Let us define the event

$$A = \{\text{the dart hits a point within 2 inches of the center}\}\,.$$

Since the dartboard as area 36π and area that makes A happen is of 4π, we have

$$\mathbb{P}(A) = \frac{4\pi}{36\pi} = \frac{1}{9}\,.$$

So we have that

$$\mathbb{P}(X=5) = \mathbb{P}(A) = \frac{1}{9}\,, \qquad \mathbb{P}(X=1) = \mathbb{P}(A^c) = 1 - \frac{1}{9} = \frac{8}{9}\,.$$

Ex. 21: We can directly apply what we have learnt in Ex. 8, that is, since the event A does not depend on the order of the sequence that we get, but just on the types of chips in the sequence, the two questions (a) and (b) will lead to the same result. So it is enough to solve the problem in just one case. We solve it in case (b) since it is easier.

Note that the sample space is made by all possible ways to draw 3 chips from $3+2+2=7$ chips, that is $\binom{7}{3} = 35$. The event A instead is made by all possible ways to draw 1 black chip, 1 red chip and 1 green chip. The number of ways to draw 1 black chip is $\binom{3}{1} = 3$. The number of ways to draw 1 red chip is $\binom{2}{1}$. Similarly, the number of ways to draw 1 green chip is $\binom{2}{1} = 2$. Hence the number of ways to draw 1 black chip, 1 red chip and 1 green chip is $\binom{3}{1}\cdot\binom{2}{1}\cdot\binom{2}{1} = 3\cdot 2\cdot 2 = 12$. So

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{12}{35}\,.$$

Ex. 23: (a) You win without changing door if the prize is behind door 1. This happens with probability $\frac{1}{3}$.

(b) Since Monty Hall opens one door, you will changing the door only in the case in which the prize is not behind the door number 1 (that implies that it will be behind the other remaining door). This happens with probability $1 - \frac{1}{3} = \frac{2}{3}$.
We deduce that it is better to change door to improve our winning probability.

Ex. 30: The chess board has 64 possible positions and hence the number of ways to choose 8 positions for the rooks is $\binom{64}{8}$. Let us now count the number of ways to choose 8 positions that don't share rows and columns.
We start by positioning the first rook in the first row: we have 8 possibilities. Fixed the first rook on the first row, we can fix the second rook in the second row: we have 7 possibilities (we have to avoid the column in which there is already the first rook). Fixed the first two rooks (that lie on the first two rows and on two different columns), we have to fix the third rook: we have 6 possibilities (since we have to avoid the two columns already occupied by the first two rooks. By iteration of the procedure, we get that the number of ways to arrange the 8 rooks so that no two chosen rooks share a row or column is

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!\,.$$

So the probability we are looking for is

$$\frac{8!}{\binom{64}{8}} = \frac{(8!)^2}{(64)_8}\,.$$

Ex. 32: First observe that the number of possible hands of poker is $\binom{52}{5}$. Let us now construct a full house (for example $3\spadesuit$, $3\heartsuit$, $3\clubsuit$, $2\spadesuit$, $2\diamondsuit$).
We have to choose the face value for the three cards: we have 13 possibilities. Then we have to choose the face value for the two cards: we have 12 possibilities (one possibility has already been used as face value for the three cards).
Now we have to choose the suits for the three cards (that have same face value). They must have three different suits chosen among 4 available suits. So the number of ways to choose them is $\binom{4}{3} = 4$.
Now we have to choose the suits for the two cards (that have same face value). They must have two different suits chosen among 4 available suits. So the number of ways to choose them is $\binom{4}{2} = 6$.
Hence the number of possible full house is given by

$$13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}\,,$$

and then we deduce that the probability of a full house is

$$\frac{13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}} \approx 0.014\,.$$

Ex. 41: Let us construct a vector (X_1, X_2, X_3, X_4) where

$$X_i = \begin{cases} 1\,, & \text{if player 1 wins the } i\text{-th game,} \\ 2\,, & \text{if player 2 wins the } i\text{-th game,} \\ 3\,, & \text{if player 3 wins the } i\text{-th game.} \end{cases}$$

So the sample space $\#\Omega$ is made by $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ elements.

Define the event $C = \{\text{at least one person wins no game}\}$ and let us define for $i = 1, 2, 3$ the event

$$A_i = \{\text{player } i \text{ wins no game}\}.$$

Note that $C = A_1 \cup A_2 \cup A_3$. By the inclusion/exclusion formula we have

$$\begin{aligned} \mathbb{P}(C) = {} & \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) \\ & - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) \\ & + \mathbb{P}(A_1 \cap A_2 \cap A_3)\,. \end{aligned} \tag{1}$$

Let us compute $\mathbb{P}(A_1)$. Note that A_1 is made by all vectors (X_1, X_2, X_3, X_4), where $X_i \in \{2, 3\}$. So $\#A_1 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$ and hence $\mathbb{P}(A_1) = \frac{16}{81}$. By symmetry $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{16}{81}$.
Note also that $A_1 \cap A_2 = \{\text{all games are won by player 3}\} = \{(3, 3, 3, 3)\}$. So $\mathbb{P}(A_1 \cap A_2) = \frac{1}{81}$. By symmetry $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3) = \frac{1}{81}$. Finally observe that $A_1 \cap A_2 \cap A_3 = \emptyset$ and hence $\mathbb{P}(A_1 \cap A_2 \cap A_3) = 0$. So by equation (1) we conclude that

$$\mathbb{P}(C) = \frac{16}{81} \cdot 3 - \frac{1}{81} \cdot 3 + 0 = \frac{5}{9}\,.$$

Ex. 43: For $n = 2$ by inclusion/exclusion formula we have

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) \leq \mathbb{P}(A_1) + \mathbb{P}(A_2)\,.$$

Let us now consider a general n and let us define $B_n = A_1 \cup \ldots \cup A_{n-1} \cup A_n$. Then applying the case $n = 2$ we have

$$\mathbb{P}(B_n) = \mathbb{P}(A_1 \cup \ldots \cup A_n) = \mathbb{P}(B_{n-1} \cup A_n) \leq \mathbb{P}(B_{n-1}) + \mathbb{P}(A_n)\,.$$

So we have

$$\mathbb{P}(B_n) \leq \mathbb{P}(B_{n-1}) + \mathbb{P}(A_n)\,.$$

We can iterate this formula for $\mathbb{P}(B_{n-1}) \leq \mathbb{P}(B_{n-2}) + \mathbb{P}(A_{n-1})$. Inserting this inequality in the above formula we get

$$\mathbb{P}(B_n) \leq \mathbb{P}(B_{n-1}) + \mathbb{P}(A_n) \leq \mathbb{P}(B_{n-2}) + \mathbb{P}(A_{n-1}) + \mathbb{P}(A_n)\,.$$

Iterating the formula we get

$$\mathbb{P}(B_n) \leqq \mathbb{P}(B_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots + \mathbb{P}(A_n)\,.$$

Noting that $B_1 = A_1$ and recalling that $B_n = A_1 \cup \ldots \cup A_{n-1} \cup A_n$, we have

$$\mathbb{P}(A_1 \cup \ldots \cup A_n) \leq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_n)\,,$$

that is what we wanted to show.