APPLIED PROBLEMS

1. Exercises

Ex. 1: N candidates are called to attend a job interview. Each candidate has been assigned, independently of the others, a day of the month of April (including Saturdays and Sundays).

(a) Calculate the probability that there are at least two candidates called on the same day.

(b) Given the names of 4 candidates, calculate the probability that all of them are called on the same day.

Ex. 2: During an exam, N students are required to leave their textbooks on the teacher's desk and can retrieve them only at the end of the test. However, the students forgot to mark their names on the books, and thus at the end each one takes a random book from those placed on the desk. Calculate the probability that no student takes their own book. Then compute such a probability when $N \to \infty$.

Ex. 3: Let us denote by $p_i \in (0, 1)$ the probability to close relay i in the electrical circuit in Figure 1, for $i = 1, 2, 3, 4, 5$. Assume that all the relays work independently of the others. Compute the probability that the current passes from A to B.

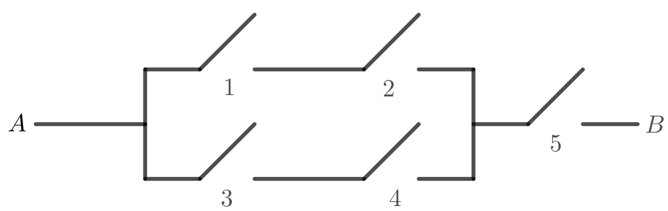

Figure 1

2. Solutions

Ex. 1: (a) Let us define the event $A = \{\text{all candidates called on different day}\}$. Note that

$$\mathbb{P}(A) = \begin{cases} \frac{(30)_N}{30^N}, & \text{if } N \leq 30 \\ 0, & \text{if } N > 30. \end{cases}$$

Indeed the number of total case (i.e. the denominator) is 30^N since each candidate can be called in a day chosen from 30 days. The numerator

instead is the number of favorable cases for A, that is the number of ways to choose the assignment of the days to all candidates in order to have no coincidences. Such a number is obtained by considering the number of ways in which we can draw N elements from 30 taking into account the order in the sample, that is $(30)_N$. So we get the above formula.
Since we want the probability of at least one match, this is $\mathbb{P}(A^c)$. So

$$\mathbb{P}(A^c) = \begin{cases} 1 - \frac{(30)_N}{30^N}\,, & \text{if } N \le 30 \\ 1\,, & \text{if } N > 30. \end{cases}$$

(b) We have fixed the four candidates. We want to compute the probability that all of them are called on the same day. Such a probability can be written as

$$\sum_{i=1}^{30} \mathbb{P}(\text{all the four candidates are called on day } i)\,.$$

Since each candidate has probability $\frac{1}{30}$ to be called on the i and since all candidates can be considered independent, we have

$$\mathbb{P}(\text{all the four candidates are called on day } i) = \left(\frac{1}{30}\right)^4\,.$$

Hence the required probability is

$$\sum_{i=1}^{30} \mathbb{P}(\text{all the four candidates are called on day } i) =$$
$$= \sum_{i=1}^{30} \left(\frac{1}{30}\right)^4 = 30 \cdot \left(\frac{1}{30}\right)^4 = \left(\frac{1}{30}\right)^3\,.$$

Ex. 2: Let us define the event

$$E_i = \{\text{the } i\text{-th student takes his own book}\}\,.$$

We would like to compute $\mathbb{P}(E_1^c \cap \ldots \cap E_N^c)$. Note that by De Morgan's law

$$\mathbb{P}(E_1^c \cap \ldots \cap E_N^c) = 1 - \mathbb{P}(E_1 \cup \ldots \cup E_N)$$

and hence we are left to compute $\mathbb{P}(E_1 \cup \ldots \cup E_N)$. Using the inclusion/exclusion formula we have

$$\begin{aligned}\mathbb{P}(E_1 \cup \ldots \cup E_N) = \sum_{i_1=1}^{N} \mathbb{P}(E_{i_1}) - \sum_{\substack{i_1,i_2=1,\ldots,N \\ i_1<i_2}} \mathbb{P}(E_{i_1} \cap E_{i_2}) + \\ + \sum_{\substack{i_1,i_2,i_3=1,\ldots,N \\ i_1<i_2<i_3}} \mathbb{P}(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \ldots + \\ + (-1)^{k+1} \sum_{\substack{i_1,i_2,\ldots,i_k=1,\ldots,N \\ i_1<i_2<\ldots<i_k}} \mathbb{P}(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) + \ldots + \\ + (-1)^{N+1} \mathbb{P}(E_1 \cap E_2 \cap \ldots \cap E_N)\,.\end{aligned}$$

Note that $E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}$ is the event that each one of the students $i_1, \ldots, i_k$ chooses his own book. To compute the probability of such an event, we can construct a vector whose i-th entry is the label of the actual original owner of the book given to student i. The number of favorable cases is given by the number of ways of constructing such a vector as

$$(1, 2, 3, \ldots, k, \underbrace{*, *, \ldots, *}_{N-k})$$

where instead of $*$ there is a random assignment of the remaining $N-k$ books. The number of ways in which we can construct such vectors is $(N-k)!$. The total number of cases in instead the number of all possible assignments of the books to the students. So $N!$. Hence

$$\mathbb{P}(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = \frac{(N-k)!}{N!}\,.$$

So we have

$$\begin{aligned}\mathbb{P}(E_1 \cup \ldots \cup E_N) = \sum_{i_1=1}^{N} \frac{(N-1)!}{N!} - \sum_{\substack{i_1,i_2=1,\ldots,N \\ i_1<i_2}} \frac{(N-2)!}{N!} + \\ + \sum_{\substack{i_1,i_2,i_3=1,\ldots,N \\ i_1<i_2<i_3}} \frac{(N-3)!}{N!} + \ldots + \\ + (-1)^{k+1} \sum_{\substack{i_1,i_2,\ldots,i_k=1,\ldots,N \\ i_1<i_2<\ldots<i_k}} \frac{(N-k)!}{N!} + \ldots + \\ + (-1)^{N+1} \frac{(N-N)!}{N!}\,.\end{aligned}$$

Since the sum $\sum_{\substack{i_1,i_2,\ldots,i_k=1,\ldots,N\\ i_1<i_2<\ldots<i_k}}$ has $\binom{N}{k}$ elements (that is the number of ways to choose k different elements from a set of N), we have

$$\begin{aligned}\mathbb{P}(E_1\cup\ldots\cup E_N) = &\binom{N}{1}\frac{(N-1)!}{N!} - \binom{N}{2}\frac{(N-2)!}{N!} + \\ &+ \binom{N}{3}\frac{(N-3)!}{N!} + \ldots + \\ &+ (-1)^{k+1}\binom{N}{k}\frac{(N-k)!}{N!} + \ldots + \\ &+ (-1)^{N+1}\frac{(N-N)!}{N!}\,.\end{aligned}$$

Recall that $0! = 1$ and note that

$$\binom{N}{k}\frac{(N-k)!}{N!} = \frac{N!}{k!\cdot(N-k)!}\cdot\frac{(N-k)!}{N!} = \frac{1}{k!}\,.$$

Hence the above probability becomes

$$\begin{aligned}\mathbb{P}(E_1\cup\ldots\cup E_N) &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \ldots + (-1)^{k+1}\frac{1}{k!} + \ldots + (-1)^{N+1}\frac{1}{N!} = \\ &= \sum_{k=1}^{N}\frac{(-1)^{k+1}}{k!}\,.\end{aligned}$$

So

$$\mathbb{P}(E_1^c\cap\ldots\cap E_N^c) = 1 - \mathbb{P}(E_1\cup\ldots\cup E_N) = 1 - \sum_{k=1}^{N}\frac{(-1)^{k+1}}{k!} = 1 + \sum_{k=1}^{N}\frac{(-1)^k}{k!}\,.$$

For $N\to\infty$ we have that by Taylor approximation

$$\sum_{i=0}^{\infty}\frac{i^k}{k!} = e^i\,,$$

and hence

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} - 1 = e^{-1} - 1\,.$$

So we have

$$\mathbb{P}(E_1^c\cap\ldots\cap E_N^c) = 1 + \sum_{k=1}^{N}\frac{(-1)^k}{k!} = 1 + e^{-1} - 1 = e^{-1}\,.$$

Ex. 3: The current passes from A to B if at least one of the following events occur

$$E = \{\text{relays 1, 2 and 5 are closed}\}\,, F = \{\text{relays 3, 4 and 5 are closed}\}\,.$$

So we have to compute $\mathbb{P}(E\cup F)$. Using the inclusion/exclusion formula we have

$$\mathbb{P}(E\cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E\cap F)\,.$$

Since relay i is closed with probability p_i independently on the others, we have that

$$\mathbb{P}(E) = p_1 p_2 p_5\,, \qquad \mathbb{P}(F) = p_3 p_4 p_5\,,$$

$$\mathbb{P}(E \cap F) = \mathbb{P}(\text{relays 1,2,3,4,5 are closed}) = p_1 p_2 p_3 p_4 p_5\,.$$

Hence

$$\mathbb{P}(E \cup F) = p_1 p_2 p_5 + p_3 p_4 p_5 - p_1 p_2 p_3 p_4 p_5 = p_5 (p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4)\,.$$