Ex. 17 (SHEET 3)

$X \sim N(-2, 7)$ $\qquad f(x) = \frac{1}{\sqrt{2\pi 7}} \cdot e^{-\frac{(x+2)^2}{2 \cdot 7}}$

$\mathbb{P}(X > 3.5) = \int_{3.5}^{+\infty} f(x)dx$ = not possible

If:
$Z \sim N(0,1)$
$\mathbb{P}(Z \le t) = \Phi(t)$

If $X \sim N(\mu, \sigma^2)$, then
$X = \sigma \cdot Z + \mu$, where $Z \sim N(0,1)$

$\mathbb{P}(X \le t) = \mathbb{P}(\sigma Z + \mu \le t) =$
$= \mathbb{P}\left(Z \le \frac{t-\mu}{\sigma}\right) = \Phi\left(\frac{t-\mu}{\sigma}\right)$

$\mathbb{P}(X > 3.5) = 1 - \mathbb{P}(X \le \underbrace{3.5}_{t}) =$
$= 1 - \Phi\left(\frac{3.5-(-2)}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{5.5}{\sqrt{7}}\right) \approx$
$\approx 1 - \Phi(2.08) \approx 1 - 0.98 = 0.02.$

$\mathbb{P}(X < -10) = \Phi\left(\frac{-10-(-2)}{\sqrt{7}}\right) = \Phi\left(\frac{-8}{\sqrt{7}}\right)$
$\approx \Phi(-3.02) =$ $\qquad \Phi(-x) = 1 - \Phi(x)$
$= 1 - \Phi(3.02) \approx 1 - 0.99 \approx 0.01$

$\mathbb{P}(-2.1 < X < -1.9) =$
$= \mathbb{P}(X < -1.9) - \mathbb{P}(X \le -2.1)$
$= \Phi\left(\frac{-1.9-(-2)}{\sqrt{7}}\right) - \Phi\left(\frac{-2.1-(-2)}{\sqrt{7}}\right)$

$= \Phi\left(\frac{0.1}{\sqrt{7}}\right) - \Phi\left(-\frac{0.1}{\sqrt{7}}\right)$
$= \Phi\left(\frac{0.1}{\sqrt{7}}\right) - \left(1 - \Phi\left(\frac{0.1}{\sqrt{7}}\right)\right) = 2\Phi\left(\frac{0.1}{\sqrt{7}}\right) - 1$
$\approx 2\Phi(0.04) - 1 \approx 2 \cdot 0.52 - 1 = 0.04$

EXERCISE SHEET 4

EX. 3

Approx. the prob. that out of 300 die rolls we get exactly 100 numbers that are multiples of 3.

$X_i = \begin{cases} 1 & \text{if the i-th die roll gives a multiple of 3} \\ 0 & \text{otherwise} \end{cases}$

We have $X_1, \dots, X_{300}$ $\Rightarrow$ they are indep.
they are identically distrib.
$\Downarrow$
i.i.d.

$S_{300} = \sum_{i=1}^{300} X_i$ = # of die rolls in which we get a multiple of 3.

$\mathbb{P}(S_{300} = 100)$

1st idea: $S_{300} = \sum_{i=1}^{300} X_i$ $\qquad X_i \sim Ber\left(\frac{2}{6}\right)$, $\frac{2}{6} = \frac{1}{3}$
$\sim Bin(300, \frac{1}{3})$

$\mathbb{P}(Bin(300, \frac{1}{3}) = 100) = \binom{300}{100} \cdot \left(\frac{1}{3}\right)^{100} \cdot \left(\frac{2}{3}\right)^{200}$

2nd idea:
if $X \sim Bin(n, p)$ and p is really small compared to n, and n is big
$\Downarrow$
$X \approx Y \sim Poi(\lambda)$ $\quad \lambda = np.$
but here $\lambda = 300 \cdot \frac{1}{3} = 100$ too big.

3rd idea: (continuity correction)
$\mathbb{P}(S_{300} = 100) = \mathbb{P}(99{,}5 \le S_{300} \le 100{,}5)$
$= \mathbb{P}(S_{300} \le 100{,}5) - \mathbb{P}(S_{300} < 99{,}5)$

CENTRAL LIMIT THEOREM
If $\{X_n\}_n$ is i.i.d with mean μ and variance σ^2 ($\mu < \infty$, $\sigma^2 < \infty$)

$\mathbb{P}\left(\frac{S_m - m\mu}{\sqrt{m\sigma^2}} \le t\right) \xrightarrow[m \to \infty]{} \Phi(t)$ $\qquad X_i \sim Ber\left(\frac{1}{3}\right)$
$\mathbb{E}[X] = 1/3$
$Var(X) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$

$\mathbb{P}(S_{300} \le 100{,}5) =$
$= \mathbb{P}\left(\frac{S_{300} - 300 \cdot \frac{1}{3}}{\sqrt{300 \cdot 2/9}} \le \frac{100{,}5 - 300 \cdot \frac{1}{3}}{\sqrt{300 \cdot 2/9}}\right)$
$= \mathbb{P}\left(\frac{S_{300} - 100}{\frac{10}{3}\sqrt{6}} \le \frac{0.5}{\frac{10}{3}\sqrt{6}}\right) \underset{CLT}{\approx} \Phi\left(\frac{0.5}{\frac{10}{3}\sqrt{6}}\right)$
$\approx \Phi(0.06) \approx 0.52.$

$\mathbb{P}(S_{300} \le 99{,}5) =$
$= \mathbb{P}\left(\frac{S_{300} - 100}{\sqrt{300 \cdot \frac{2}{9}}} \le \frac{99{,}5 - 100}{\sqrt{300 \cdot 2/9}}\right) \overset{CLT}{\approx} \Phi\left(\frac{-0.5}{\frac{10}{3}\sqrt{6}}\right)$
$= 1 - \Phi\left(\frac{0.5}{\frac{10}{3}\sqrt{6}}\right) \approx 1 - \Phi(0.06) = 1 - 0.52$

$\mathbb{P}(S_{300} = 100) \approx 0.52 - (1 - 0.52) = 0.04.$

Ex. 5

Liz is standing on the real number line at position 0.

She rolls a die repeatedly.
If the roll is 1 or 2 $\Rightarrow$ she takes 1 step to the right
If the roll is 3,4,5,6 $\Rightarrow$ she takes 2 steps to the right
Define X_n the position of Liz after n rounds.

a) $\lim_{n \to \infty} \mathbb{P}(X_n > 1.6\,n)$

$Y_i = \begin{cases} 1 & \text{if at the ith roll we get 1 or 2} \\ 2 & \text{otherwise} \end{cases}$

$X_n = \sum_{i=1}^{n} Y_i$

$\{Y_i\}_i$ are i.i.d.

$\mathbb{P}(X_n > 1.6\,n) = \mathbb{P}\left(\frac{\sum_{i=1}^{n} Y_i - n \cdot \mathbb{E}[Y_i]}{\sqrt{n Var(Y_i)}} > \frac{1.6n - n\mathbb{E}[Y_i]}{\sqrt{n Var(Y_i)}}\right)$

$\mathbb{E}[Y_i] = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$ $\qquad \mathbb{E}[Y_i^2] = 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{2}{3} = 3.$

$Y_i = \begin{cases} 1 & p = 1/3 \\ 2 & p = 2/3 \end{cases}$ $\qquad Z_i \sim Ber(2/3)$ $\qquad Y_i = 1 + Z_i$

$Var(Y_i) = \mathbb{E}[Y_i^2] - \mathbb{E}[Y_i]^2 = 3 - \frac{25}{9} = \frac{2}{9}$

$\mathbb{P}\left(\frac{X_n - n \cdot 5/3}{\sqrt{n \cdot 2/9}} > \frac{1.6n - n \cdot 5/3}{\sqrt{n \cdot 2/9}}\right) \underset{\text{CLT for n large}}{\approx} 1 - \Phi\left(\frac{\sqrt{n}(1.6 - 5/3)}{\frac{\sqrt{2}}{3}}\right)$

$\xrightarrow[n \to +\infty]{} 1 - \underbrace{\lim_{x \to -\infty} \Phi(x)}_{0} = 1$ (CDF)

b) $\mathbb{P}(X_n > 1.7\,n) \underset{CLT}{\approx} 1 - \Phi\left(\frac{\sqrt{n}(1.7 - 5/3)}{\frac{\sqrt{2}}{3}}\right)$
$\xrightarrow[n \to +\infty]{} 1 - \underbrace{\lim_{x \to +\infty} \Phi(x)}_{1} = 0$

Ex. 11

On the first 300 pages you notice that there are, on average, 6 typos per page. What is the prob. that there will be at least 4 typos on page 301?

Each letter in the page is an experiment that can be a success (that is, it is wrong) with very small prob.
The number of letters in a page is huge.
The experiments are i.i.d.

X = # of typos in page 301
$X \sim Poi(\lambda)$, $\lambda = 6$
$\mathbb{P}(X \ge 4) = 1 - \mathbb{P}(X < 4)$

X assumes values $0, 1, 2, 3, \dots$
$= 1 - [\underbrace{\mathbb{P}(X=0)}_{=e^{-6}} + \underbrace{\mathbb{P}(X=1)}_{=6e^{-6}} + \underbrace{\mathbb{P}(X=2)}_{=18e^{-6}} + \underbrace{\mathbb{P}(X=3)}_{36e^{-6}}]$

$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$ $\qquad 0! = 1$

$= 1 - 61e^{-6}.$

Ex. 20

You flip a fair coin 10.000 times. Approx. the prob. that the difference between the n° of H and T is at most 100.

$X_i = \begin{cases} 1 & \text{if the i-th round is H} \\ 0 & \text{otherwise} \end{cases}$

$S_{10000} = \sum_{i=1}^{10000} X_i$ = # H in 10000 rounds.

$|H - T| = |\underbrace{S_{10000}}_{H} - \underbrace{(10.000 - S_{10.000})}_{T}|$
$= |2S_{10000} - 10.000|$ $\qquad |x| \le a \Rightarrow -a \le x \le a$

$\mathbb{P}(|2S_{10000} - 10.000| \le 100)$
$= \mathbb{P}(-100 \le 2S_{100} - 10.000 \le 100)$
$= \mathbb{P}\left(\frac{10.000 - 100}{2} \le S_{100} \le \frac{10.000 + 100}{2}\right)$
$= \mathbb{P}(4950 \le S_{100} \le 5\,050)$

$\mathbb{E}[X_i] = 1/2$ $\qquad X \sim Ber(p)$
$Var(X_i) = 1/2 \cdot 1/2 = 1/4$ $\qquad \mathbb{E}[X] = p$
$\qquad Var(X) = p(1-p)$

$\mathbb{P}\left(\frac{4950 - 10000 \cdot 1/2}{\sqrt{10000 \cdot 1/4}} \le \frac{S_{100} - 10000 \cdot 1/2}{\sqrt{10000 \cdot 1/4}} \le \frac{5050 - 10000 \cdot 1/2}{\sqrt{10000 \cdot 1/4}}\right)$
$= \mathbb{P}\left(\frac{-50}{50} \le \frac{S_{100} - 10000 \cdot 1/2}{\sqrt{10000 \cdot 1/4}} \le \frac{50}{50}\right)$
$\underset{CLT}{\approx} \Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1))$
$= 2\Phi(1) - 1 \approx 2 \cdot 0.84 - 1 = 0.68.$

Ex. 12

$T \sim Exp(\lambda)$ $\qquad$ p.d.f.: $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}[T] = 1/\lambda$ $\quad Var(T) = 1/\lambda^2$

$\mathbb{E}[T^3] = \int_{-\infty}^{+\infty} x^3 f(x)dx = \int_0^{+\infty} \underbrace{x^3}_{g} \cdot \underbrace{\lambda e^{-\lambda x}}_{f'} dx =$

$= \underbrace{\left[x^3 \cdot (-e^{-\lambda x})\right]_0^{+\infty}}_{0} - \int_0^{+\infty} 3x^2(-e^{-\lambda x})dx$ $\qquad f(x) = \lambda \frac{e^{-\lambda x}}{-\lambda} = -e^{-\lambda x}$, $g'(x) = 3x^2$

$= 3\int_0^{+\infty} x^2 e^{-\lambda x} dx = \frac{3}{\lambda} \int_0^{+\infty} x^2 \cdot \lambda e^{-\lambda x} dx$

$= \frac{3}{\lambda} \underbrace{\int_{-\infty}^{+\infty} x^2 f(x) dx}_{\mathbb{E}[T^2]} = \frac{3}{\lambda} \mathbb{E}[T^2] = \frac{6}{\lambda^3}.$

$\mathbb{E}[T^2] = Var(T) + \mathbb{E}[T]^2 = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}$