Two main results in Probability

a) Law of large numbers $E[X] < \infty$

b) Central Limit theorem $E[X^2] < \infty$

Both require samples, an iid sequence

$X_1, X_2, X_3, \dots, X_n$

$S_n = \sum_{i=1}^{n} X_i \qquad E[S_n] = n\mu \qquad Var(S_n) = n\sigma^2$

$\frac{S_n}{n}$ average of $X_1, X_2, \dots$ identically distributed

Problem we might want to do a experiment, and we want to estimate μ, $\mu = \frac{S_n}{n}$

We need $E[\hat{\mu}] = \mu$ (unbiasedness), $\hat{\mu} \to \mu$ in probability

$\lim_{n\to\infty} E[\hat{\mu}] = \mu$ Good estimator!

and $Var(\hat{\mu}) \to 0$ (consistency, $\hat{\mu} \to \mu$ in probability)

$P(|\hat{\mu} - \mu| > \epsilon) \to 0$

$P(|\hat{\mu} - \mu| > \epsilon) \le \frac{Var(\hat{\mu})}{\epsilon^2}$ $\quad \epsilon > 0$

$Var(\hat{\mu}) \to 0$ as $n \to \infty$ $\quad P(|\hat{\mu} - \mu| > \epsilon) \to 0$

$Var(\hat{\mu}) = Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2} Var(S_n) = \frac{\sigma^2}{n}$

$E[S_n] = n\mu$ $\quad$ $S_n = \sum_{i=1}^{n} X_i$ $\quad$ X_1, X_2, X_3 are common mean and variance

$\frac{S_n}{\sqrt{n}} - \mu$ = probability $\sqrt{n}\left(\frac{S_n}{n} - \mu\right)$ in prob

$\frac{S_n}{\sqrt{n}} - \mu$ = Distribution

Bernoulli $X_i \sim Ber(p)$

$\frac{S_n}{\sqrt{n}} - \sqrt{n}\mu$

$Var\left(\frac{S_n}{\sqrt{n}} - \sqrt{n}\mu\right) = Var\left(\frac{S_n}{\sqrt{n}}\right) = \frac{\sigma^2}{n} = \sigma^2$

$Var(\sqrt{n}\bar{X}_n - \sqrt{n}\mu) = n Var(\bar{X}_n) = n \frac{\sigma^2}{n} = \sigma^2$ constant as $n \to \infty$

$\sqrt{n}(\bar{X}_n - \mu)$ $\quad \bar{X}_n = \frac{1}{n}\sum X_i$ $\quad \frac{1}{n}\sum (X_i - \mu) = \frac{1}{n}\sum X_i - n\mu$

Study that $\sqrt{n}(\bar{X}_n - \mu)$

$\sqrt{n}(\bar{X}_n - \mu) = \sqrt{n}\left(\frac{1}{n}\sum X_i - \mu\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu)$

$\frac{1}{n}\sum_{i=1}^{n} X_i - \mu = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)$

$$\boxed{\sqrt{n}(\bar{X}_n - \mu) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}(X_i - \mu)\right)}$$

Mean zero, variance 1 $\quad \sigma^2 = p(1-p)$

$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}(X_i - \mu)\right)$ $\quad X_1, X_2, X_3 \dots \sim Ber(p)$

Binomial $\quad S_n = X_1 + X_2 + \dots + X_n \sim Bin(n, p)$

$$\forall t \quad \lim_{n\to\infty} P\left(\frac{S_n - np}{\sqrt{np(1-p)}} \le t\right) = \Phi(t)$$

CENTRAL LIMIT THEOREM $\quad$ CDF of $N(0,1)$

$S_n \sim B(n,p) \qquad P(a \le S_n \le b) = \sum_{k=a}^{b} \binom{n}{k} p^k (1-p)^{n-k}$

$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$

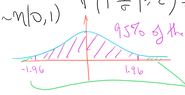
$$P\left(\frac{a-np}{\sqrt{np(1-p)}} \le \frac{S_n - np}{\sqrt{np(1-p)}} \le \frac{b-np}{\sqrt{np(1-p)}}\right) = \Phi\left(\frac{b-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a-np}{\sqrt{np(1-p)}}\right)$$

$$P(|X - \mu| \ge t) \le \frac{Var(X)}{t^2}$$

$$P(|X - \mu| \ge 2\sigma) \le \frac{Var(X)}{(2\sigma)^2} = \frac{\sigma^2}{4\sigma^2} = \frac{1}{4}$$

$X \sim N(\mu, \sigma^2)$

$\frac{X - \mu}{\sigma} \sim N(0,1) \qquad P\left(\left|\frac{X-\mu}{\sigma}\right| \ge 2\right) = P(|Z| \ge 2)$



$\frac{1}{20}$

$t > 0, \quad Z \sim N(0,1) \qquad t \le x < \infty$

$$P(Z > t) = \int_t^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \qquad 1 \le \frac{x}{t}$$

$0 \le g(x) \le f(x)$

$\boxed{f(x) \le g(x)}$

$\int_a^b f(x) \le \int_a^b g(x) dx$

$$\le \int_t^{\infty} \frac{1}{\sqrt{2\pi}} \frac{x}{t} e^{-x^2/2} dx = \frac{1}{t\sqrt{2\pi}} \int_t^{\infty} x e^{-x^2/2} dx$$

$$= \frac{1}{t\sqrt{2\pi}} \left(e^{-x^2/2}\right)\Big|_{\infty}^{t} = \frac{e^{-t^2/2}}{t\sqrt{2\pi}}$$

$$P(Z > 2) \le \frac{e^{-2}}{2\sqrt{2\pi}}$$

R pnorm

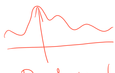
De Moivre 1733

Laplace 1810 Generalization

C.L.T, $X_1, X_2, \dots$ iid

with mean μ, and variance σ^2

$S_n = X_1 + \dots + X_n$ $\quad$ then $\quad \lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le t\right) = \Phi(t) \quad \forall t \in \mathbb{R}$



Proof via transforms

Accuracy

$\frac{S_n}{n}$ estimate μ

of the CLT is

the topic of the Berry-Esseen Theorem

Accuracy $\le$ constant $\times$ third moment $E|X|^3$ / standard deviation, n

0.4...

$\frac{S_n}{n} \to \mu$ in prob as $n \to \infty$

ϵ tolerance

α level

$P(\text{good}) \ge \alpha$

$\alpha = 0.95$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \le \epsilon\right) = P\left(-\epsilon \le \frac{S_n}{n} - \mu \le \epsilon\right)$$

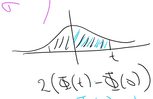
$$= P\left(-\epsilon \le \frac{S_n - n\mu}{n} \le \epsilon\right)$$

$$= P\left(-\sqrt{n}\epsilon \le \frac{S_n - n\mu}{\sqrt{n}} \le \sqrt{n}\epsilon\right)$$

$$= P\left(-\frac{\sqrt{n}\epsilon}{\sigma} \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le \frac{\sqrt{n}\epsilon}{\sigma}\right)$$

$$\cong P\left(-\frac{\sqrt{n}\epsilon}{\sigma} \le Z \le \frac{\sqrt{n}\epsilon}{\sigma}\right)$$

$$= \Phi\left(\frac{\sqrt{n}\epsilon}{\sigma}\right) - \Phi\left(-\frac{\sqrt{n}\epsilon}{\sigma}\right)$$



$2(\Phi(t) - \Phi(0))$

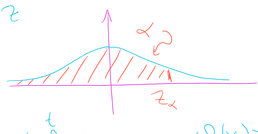
$= 2\Phi(t) - 1$

$$= 2\Phi\left(\frac{\sqrt{n}\epsilon}{\sigma}\right) - 1$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \le \epsilon\right) \cong 2\Phi\left(\frac{\sqrt{n}\epsilon}{\sigma}\right) - 1 \ge \alpha$$

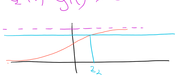
$$\Phi\left(\frac{\sqrt{n}\epsilon}{\sigma}\right) \ge \frac{1+\alpha}{2} = \Phi\left(z_{\frac{1+\alpha}{2}}\right)$$

z



$\Phi(z_\alpha) = \alpha \qquad \frac{\sqrt{n}\epsilon}{\sigma} = z_{(1+\alpha)/2}$

$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx \qquad g(x) > 0 \ \forall x$

$\Phi'(t) = g(t) > 0$ Strictly increasing

$0 < \alpha < 1$