$$\mathrm{Var}(k) \ge \frac{\mathrm{Cov}(k)}{\mathrm{Var}(k)} \quad \frac{\mathrm{Cov}(k)}{\mathrm{Var}(k)} \ge 0$$

$$\mathrm{Var}(k) \ge \frac{\mathrm{Cov}(k)}{\mathrm{Var}(k)} - \frac{2\,\mathrm{Cov}(k)}{\mathrm{Var}(k)} > 0$$

$$\mathrm{Var}(k) \ge \frac{\mathrm{Cov}(k)}{\mathrm{Var}(k)}$$

$$\boxed{\mathrm{Cov}(k) \le \mathrm{Var}(k)\,\mathrm{Var}(k)}$$ Cauchy Schwarz

$$\frac{\mathrm{Cov}(k)}{\mathrm{Var}(k)\,\mathrm{Var}(k)} = 1 \quad \text{w/ } \xi = 1$$

$$-1 \le \xi \le 1$$

Markov's Inequality

$X \ge 0$, $t > 0$ (Bernoulli)

$\mathbb{E}[1_{\{X \ge t\}}] = P(X \ge t)$

$\mathbb{E}[1_{\{X \ge t\}}] \le P(X \ge t)$

$P(X \ge t) = \mathbb{E}[1_{\{X \ge t\}}] = \mathbb{E}\left[\frac{t}{t} 1_{\{X \ge t\}}\right] = \mathbb{E}\left[\frac{X}{t}\right] = \frac{\mathbb{E}[X]}{t}$

Claim $1_{\{X \ge t\}} \le \frac{X}{t} 1_{\{X \ge t\}} \le \frac{X}{t}$

$X \ge 0$, $t > 0$

$$P(X \ge t) \le \frac{\mathbb{E}[X]}{t}$$

$$P(|X-\mu| \ge t) = P(\underbrace{(X-\mu)^2}_{} \ge \underbrace{t^2}_{}) \le \frac{\mathbb{E}[(X-\mu)^2]}{t^2} = \frac{\mathrm{Var}(X)}{t^2}$$

$$\boxed{P(|X-\mu| \ge t) \le \frac{\mathrm{Var}(X)}{t^2}}$$ Chebyshev's Inequality

tail probability

Poisson Distribution $\lambda > 0$

$X \in \{0,1,2,\ldots\}$

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad \mathbb{E}[X] = \sum_{k=0}^{\infty} k\, e^{-\lambda}\frac{\lambda^k}{k!}$$

X satisfies Stein Identity $\quad \mathbb{E}[X g(X)] = \lambda\,\mathbb{E}[g(X+1)] \quad \forall g$

$\forall k$ $\quad g(k) = 1$, $\quad \mathbb{E}[X] = \lambda$

$g(k) = k \quad \mathbb{E}[X^2] = \lambda\,\mathbb{E}[X+1] = \lambda(\mathbb{E}[X]+1) = \lambda^2 + \lambda$

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Verify: $\quad \mathbb{E}[X g(X)] = \sum_{k=0}^{\infty} k g(k)\, e^{-\lambda}\frac{\lambda^k}{k!}$

$$= \sum_{k=1}^{\infty} k g(k)\, e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} g(k)\, e^{-\lambda}\frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=1}^{\infty} g(k)\, e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}$$

$j = k-1$

$$= \lambda \sum_{k=0}^{\infty} g(k+1)\, e^{-\lambda}\frac{\lambda^k}{k!} = \lambda\,\mathbb{E}[g(X+1)]$$

If $X \sim \mathrm{Pois}(\lambda) \iff \mathbb{E}[X g(X)] = \lambda\,\mathbb{E}[g(X+1)] \quad \forall g$

Verify $\quad g(X) = 1_{(X=k)}$ $\qquad$ $g(X+1) = 1_{(X+1=k)} = 1_{(X=k-1)}$

LHS $\qquad$ RHS

$$\mathbb{E}[X 1_{(X=k)}] = \mathbb{E}[k 1_{(X=k)}]$$

$$\lambda\,\mathbb{E}[1_{(X=k-1)}] = \lambda P(X=k-1)$$

$$X 1_{(X=k)} = k 1_{(X=k)} \qquad = k P(X=k) \qquad k P(X=k) = \lambda P(X=k-1)$$

$$\boxed{P(X=k) = \frac{\lambda}{k} P(X=k-1)} \Rightarrow P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Large $\mathrm{Bin}(n,p)$ small approx Poisson

$X_n \sim \mathrm{Bin}(n, \lambda/n)$ $\quad Y \sim \mathrm{Pois}(\lambda)$ $\quad$ counting the number of occurrences of rare events

$P(X_n = k) \to P(Y = k)$

n trials, d steps

X = # successes

$X_n = 1_{\{\text{success at kth step}\}}$ Bernoulli $\mathbb{E}[X_n] = \frac{1}{n}$

$X = \sum_{k=1}^{n} X_n$, $\quad \mathbb{E}[X] = \sum_{k=1}^{n} \frac{1}{n} = \lambda$

$X \le \mathrm{Bin}(n)$ $\quad P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

approx

$P(X=0) = e^{-\lambda} =$

$n = 50$ $\quad \lambda = \frac{1}{50} \cdot 50 = 0.6931 > 0.1 \cdot \ln 2$

$e^{-\lambda} < e^{-\ln 2} = \frac{1}{2}$

$P(X \ge 2) = 1 - P(X=0) - P(X=1)$

$= 1 - e^{-\lambda} - \lambda e^{-\lambda}$ $\quad n = 144$ $\quad \lambda_{144} = \left(\frac{1}{2}\right)\frac{1}{50} = 2.59$

$P_n(X \ge 2) = 0.73$

Gaussian Distribution

$$g(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \qquad \Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

no "closed form" for $\Phi(t)$

a) $g(t) \ge 0$ ✓

b) $\int_{-\infty}^{\infty} g(t)\,dt = 1$

$$\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$

$$\int_a^b\int_c^d h(x) g(y)\,dx\,dy = \int_c^d g(y)\left(\int_a^b h(x)\,dx\right)dy = \int_a^b h(x)\,dx \cdot \int_c^d g(y)\,dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$t = s^2$

$\frac{dt}{ds} = 2s\,ds$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}}\, r\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{dx}{dr} = \cos\theta$

$$\begin{bmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$(\cos\theta)(r\cos\theta) - (-r\sin\theta)\sin\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta$$

$$\frac{d}{dr}\left(e^{-r^2/2}\right) = -e^{-r^2/2}\, r$$

flip limits

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(e^{-r^2/2}\Big|_{\infty}^{0}\right)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

Use Z for variables with density $g(t)$

location/scale family

μ

$\sigma > 0 \quad \mu \in \mathbb{R}$

Let $X = \sigma Z + \mu$

even

$g(t) = g(-t)$

$\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

p.d.f.

CDF

$$P(X \le t) = P(\sigma Z + \mu \le t)$$

$$= P(\sigma Z \le t - \mu) = P\left(Z \le \frac{t-\mu}{\sigma}\right) = F_Z\left(\frac{t-\mu}{\sigma}\right)$$

$$P(X \le t) = \Phi\left(\frac{t-\mu}{\sigma}\right) \qquad \Phi(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$$

$$\Phi'(s) = g(s) \text{ FTOC}$$

Density

$$g_{\mu,\sigma}(t) = \frac{1}{\sigma} g\left(\frac{t-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

$\mathcal{N}(\mu, \sigma^2)$