$B \sim U[0,1]$

side length between 0 and 1

Find CDF, pdf of X if side length is random point.

$P(X \le t)$

$\{x \le t\}$ — x random & $x \le t$ / random point in first half — B

$= \{(x \le t_2) \cup (t_2 \le x \le t)\}$ — $t_2 = t/2$

$= \{B \le t_2\} \cup \{t_2 \le B \le t\}$ — $t_2 = t/2$, $t_2 = t$

$P(x \le t) = P(B \le t/2) + P(t/2 \le B \le t)$

$x \sim U[0,1]$

$0 \le t \le 1/2$

Case

$P(X \le t) = \{B \le t\} \cup \{B \ge 1-t\}$

$P(X \le t) = P(B \le t) + P(B \ge 1-t)$

$= 2P(B \le t)$

$= 2t/2$ $\quad 0 \le t \le 1/2$

$F(t) = \begin{cases} 2t & 0 \le t \le 1/2 \\ 1 & t \ge 1/2 \end{cases}$

Case $1/2 \le t \le 1$

Suppose stick broken at 2 random points independently

$x \sim U[0,1]$

$y \sim U[0,1]$

(U, V) pair

joint distribution

Find U, V with all the three conditions in example.

T: Being form a triangle

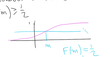
Q: What T as an event involving U, V & U_2

X "center"? $\mathbb{E}[X]$ expectation

median, m: a number such that

$P(X \ge m) \ge \frac{1}{2}$ $\quad P(X \le m) \ge \frac{1}{2}$

If F is strictly increasing



$F(m) = \frac{1}{2}$

pth quantile, x

$P(X \le x) \ge 1-p$, $P(X \ge x) \ge p$

x is pth quantile

x is 90th percentile

median

center $\mathbb{E}[X]$

spread

Variance, Standard deviation

how wide does X spread from its mean $\mu = \mathbb{E}[X]$

$X - \mu$: difference from the mean

$|X - \mu|$ absolute difference

$(X - \mu)^2$ squared difference

$\mathbb{E}|X - \mu|$ expected absolute difference

$\mathbb{E}(X - \mu)^2$: expected squared difference, mean Squared Error

Variance $Var(X) = \mathbb{E}(X - \mu)^2$

$\mathbb{E}(X - \mu)^2 = \mathbb{E}(X^2 - 2\mu X + \mu^2)$ $\quad \int (x-\mu)^2 f(x)\,dx$

Linearity of expectation: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$, $a, b \in \mathbb{R}$

$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$

$= \mathbb{E}[X^2] + \mathbb{E}[-2\mu X] + \mathbb{E}[\mu^2]$

$= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2$

Second moment

$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 = \mathbb{E}[X^2] - \mu^2$

$= \mathbb{E}(X) - (\mathbb{E}(X))^2$

X: approx spread by σ on average

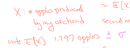
second moment minus square of 1st moment

order $\mathbb{E}(X)$, 1.797 approx $\pm\ \sigma$

$\sigma^2 = Var[X]$, 3.226, 1.413 approx

$\sigma = \sqrt{Var[X]}$

$X \sim U[a,b]$ $\quad \mathbb{E}[X] = \frac{b+a}{2}$ 1st moment



$\mathbb{E}[X] = \int_a^b f(x)x\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$

$= \frac{1}{(b-a)}\left(\frac{b^2 - a^2}{2}\right) = \frac{1}{2(b-a)}((b-a)(b+a))$

$\mathbb{E}(X) = \frac{b+a}{2}$

$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$

$X \sim P(\lambda)$, $X \in \{0, 1, 2, \ldots\}$, $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

$\mathbb{E}(X) = \lambda$, $\mathbb{E}[X] = \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty}\frac{k\lambda^k}{(k-1)!}$

$= e^{-\lambda}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}k\lambda^k = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\lambda^{k-1}$ $\quad e^{\lambda} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$

$= \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\frac{1}{\partial\lambda}\lambda^k = \lambda e^{-\lambda}\frac{1}{\partial\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}$

$= \lambda e^{-\lambda}\frac{1}{\partial\lambda}\left(\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}\right) = \lambda e^{-\lambda}\frac{1}{\partial\lambda}\left(\lambda\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\right)$

$j = k-1$

$k=1 \Rightarrow j=0$

$= \lambda e^{-\lambda}\frac{1}{\partial\lambda}\left(\lambda e^{\lambda}\right)$

$= \lambda e^{-\lambda}\left(e^{\lambda} + \lambda e^{\lambda}\right) = \lambda(1 + \lambda) = \lambda^2 + \lambda$

$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Same as the mean.

Linearity of Expectation

$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$

$X \sim Bin(n, p)$, $\mathbb{E}[X] = \sum_{k=0}^{n} k^2 P(X = k)$ $\quad \mathbb{E}[X^2]$

$= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$

cancelled k, rewrote the sum, pulled term out, used the Binomial theorem

Obtain $\mathbb{E}[X] = np$

$B \sim Ber(p)$, $\mathbb{E}[B] = 1 \cdot P(B = 1) + 0 \cdot P(B = 0) = p$

$B \in \{0, 1\}$ $\quad P(B = 0) = 1 - p$ $\quad P(B = 1) = p$

Indicator variables $\quad I_A \in \{0, 1\}$

only

100111 $\quad I_A = \begin{cases} 1 & \text{Heads on toss } 1 \\ 0 & \text{otherwise} \end{cases}$

$X = 4$

$I_1, I_2, I_3, I_4, I_5, I_6$ $\quad P(I_1 = 1) = p$, $\mathbb{E}[I_1]$

$X = I_1 + I_2 + \ldots + I_n$ $\quad X \sim B(n, p)$ $\quad \mathbb{E}[I_1] = p$ all n

$Var(I_1) = p - p^2 = p(1-p)$

$\mathbb{E}[X] = \mathbb{E}[I_1] + \ldots + \mathbb{E}[I_n] = np$

$\mathbb{E}(X^2) - (\mathbb{E}(X))$

If heavily hints for Variance in $B(n,p)$ $\quad Var(X) = nVar(I_1) = np(1-p)$

$S_n = X_1 + \ldots + X_n$

$Var(S_n) = Var(X_1) + \ldots + Var(X_n)$?

$\mathbb{E}[X] = \mu$, $\mathbb{E}[Y] = \nu$

$\mathbb{E}[X + Y] = \mu + \nu$ $\quad \mathbb{E}(U - \mathbb{E}[U])^2$

$Var(X + Y) = \mathbb{E}[(X + Y - \mathbb{E}(X + Y))^2]$

$= \mathbb{E}[(X + Y - \mathbb{E}X - \mathbb{E}Y)^2]$

$= \mathbb{E}[(X - \mathbb{E}X) + (Y - \mathbb{E}Y)]$

$= \mathbb{E}(X - \mathbb{E}X)^2 + \mathbb{E}(Y - \mathbb{E}Y)^2 + 2\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$ $\quad$ bdy $\quad$ $\mathbb{E}(Y)$ $\quad$ $X \perp\!\!\!\perp Y \Rightarrow X \perp Y$

$Var(X + Y) = Var(X) + Var(Y) + 2\,Cov(X, Y)$ $\quad$ If $\mu = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$

$= \mathbb{E}[(X - \mathbb{E}X)] \cdot \mathbb{E}[Y - \mathbb{E}Y]$

$Var(X + Y) = Var(X) + Var(Y)$

If $X \perp\!\!\!\perp Y$ $\quad Cov(X, Y) = 0$

$\mathbb{E}(X - \mathbb{E}X) = \mathbb{E}X - \mathbb{E}(\mathbb{E}X)$

$= \mathbb{E}X - \mathbb{E}X = 0$

$X \perp\!\!\!\perp Y \Rightarrow Cov(X, Y) = 0$ $\quad$ uncorrelated

$\forall x, y$ $P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$

Also, $\mathbb{E}[h(X)g(Y)] = \mathbb{E}(h(X)) \cdot \mathbb{E}[g(Y)]$

$X \sim Bin(n, p)$ $\quad I_1, I_n$ mutually independent

$X = \sum_{i=1}^{n} I_i$ $\quad \mathbb{E}[X] = \sum_{i=1}^{n}\mathbb{E}[I_i] = \sum_{i=1}^{n} p = np$

$Var(X) = \sum_{i=1}^{n} Var(I_i) = \sum_{i=1}^{n} p(1-p) = np(1-p)$

Covariance $Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$

$Cov(aX + b, cY + d) = ac\,Cov(X, Y)$

$\rho = Cov(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$ $\quad Var(X) = 0$ $\quad P(X = c) = 1$ for some c

$\boxed{Cov(aX + b, cY + d) = Cov(X, Y)}$

Fact: $\rho \in [-1, 1]$ $\quad |\rho| = 1$ perfect correlation, means X or Y is a linear function of the other

Find X, Y with that $Cov(X, Y) = 0$

but $Y = h(X)$